Exact black hole solutions in higher-order scalar-tensor theories

Nicolas Lecœur

Université Paris-Saclay, CNRS/IN2P3, IJCLab, 91405 Orsay, France

based on E. Babichev, C. Charmousis and N. L., [arXiv:2309.12229 [gr-qc]]

November 6

Outline

Introduction: scalar-tensor theories

Stealth black holes and their conformal-disformal transformations

Non-stealth black holes: parity and shift-symmetric theories

Non-stealth black holes: generic Horndeski theories

Conclusions

Scalar-tensor theories

- Modifying General Relativity is motivated from both theoretical and observational considerations [E. J. Copeland, M. Sami, S. Tsujikawa, 2006] [T. Clifton, P. G. Ferreira, A. Padilla, C. Skordis, 2012]
- Lovelock's theorem: the unique 4D action depending on the metric only and yielding second-order conserved field equations is the Einstein-Hilbert action with cosmological constant
- Scalar-tensor gravity: modification of gravity which includes, in addition to the usual metric **tensor** field $g_{\mu\nu}$, a **non-minimally coupled scalar field** ϕ
- Adds a unique degree of freedom $\rightsquigarrow$ both simple and general [T. Chiba, Phys.Lett.B, 2003]

(Hairy) black holes

Focus: **asymptotically flat black holes in vacuum without Λ**

In GR, stationary black holes are completely characterized by mass M and angular momentum J (Kerr metric). They have **no hair**, i.e., no independent integration constant other than M and J

In scalar-tensor gravity, **a black hole is said hairy if it is dressed with a non-trivial scalar field**: $\phi \neq 0$. A hairy black hole can be:

- **Stealth**: the metric is the same as in GR
- **Non-stealth**: the metric is different from GR. The hair is **secondary** if the metric is still determined only by M and J, and **primary** if the metric depends on another integration constant, independent from M and J

Exact black hole solutions here means **closed-form**: no numerical integration, no expansion in the couplings, ...

Horndeski theory

[Horndeski, 1974]

Most general 4D action with **second-order field equations**. **Four arbitrary functions** $G_k(\phi, X)$, $k = 2, 3, 4, 5$:

$$S[g_{\mu\nu}, \phi] = \int \mathrm{d}^4x\sqrt{-g}\left\{\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5\right\},$$

$$\mathcal{L}_2 = G_2, \quad \mathcal{L}_3 = -G_3\Box\phi, \quad \mathcal{L}_4 = G_4 R + G_{4X}\left[(\Box\phi)^2 - (\phi_{\mu\nu})^2\right],$$

$$\mathcal{L}_5 = G_5\, G_{\mu\nu}\phi^{\mu\nu} - \frac{1}{6}G_{5X}\left((\Box\phi)^3 - 3\Box\phi\,(\phi_{\mu\nu})^2 + 2\phi_{\mu\nu}\phi^{\nu\rho}\phi^{\mu}_{\rho}\right)$$

X is the **scalar field kinetic term**, $X = -(\partial\phi)^2/2$

If $G_k = G_k(X)$ for all k, **shift symmetry** under shifts of the scalar $\phi \to \phi + \text{const}$ $\rightsquigarrow$ conserved Noether current, $J^\mu = -\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta(\partial_\mu\phi)}$

If only $G_2(X)$ and $G_4(X)$, **parity symmetry** under $\phi \to -\phi$

Degenerate Higher-Order Scalar-Tensor (DHOST) theories

[D. Langlois, K. Noui, 2015]

Field equations of order > 2 but evading the Ostrogradsky instability thanks to degeneracy of the kinetic matrix

Disformal transformation: $\widetilde{g}_{\mu\nu} = C\left(\phi, X\right) g_{\mu\nu} + D\left(\phi, X\right) \partial_\mu \phi\, \partial_\nu \phi$

If $S \in$ DHOST, then $\widetilde{S} \in$ DHOST where $S\left[g_{\mu\nu}, \phi\right] = \widetilde{S}\left[\widetilde{g}_{\mu\nu}, \phi\right]$

$\rightsquigarrow$ **Generation of solutions**: starting from a *seed solution* $\left(g_{\mu\nu}, \phi\right)$ to the initial action S, get a solution $\left(\widetilde{g}_{\mu\nu}, \phi\right)$ to the new action $\widetilde{S}$

If $S \in$ Horndeski, $C = C\left(\phi\right)$ and $D = D\left(\phi, X\right)$, then $\widetilde{S} \in$ **beyond Horndeski**:

$$S_{\mathrm{bH}} = \int \mathrm{d}^4 x \sqrt{-g} \left\{ \mathcal{L}_{4\mathrm{bH}} + \mathcal{L}_{5\mathrm{bH}} \right\},$$

$$\mathcal{L}_{4\mathrm{bH}} = F_4\left(\phi, X\right) \varepsilon^{\mu\nu\rho\sigma}\, \varepsilon^{\alpha\beta\gamma}{}_{\sigma}\, \phi_\mu \phi_\alpha \phi_{\nu\beta} \phi_{\rho\gamma}, \quad \mathcal{L}_{5\mathrm{bH}} = F_5\left(\phi, X\right) \cdots$$

$\phi = \cdots$	$G_{2,4}(X)$ (shift + parity sym. Horndeski)	$G_{2,3,4,5}(\phi, X)$ (Generic Horndeski)	$G_{2,4}(X)$ and $F_4(X)$ (shift + parity sym. beyond Horndeski)	DHOST
$\phi(r)$	BCL (Babichev-Charmousis-Lehébel)	4DEGB (Einstein-Gauss-Bonnet) + extensions without conformal invariance	Secondary hair solutions, homogeneous or not	Regular black holes (Kerr-Schild construction)
$qt+\psi(r)$ and $X =$ constant	Stealth Schwarzschild	$\emptyset$	$\emptyset$	Stealth Kerr and Disformed Kerr
$qt+\psi(r)$ and $X \neq$ constant	(New) Stealth Schwarzschild	Shift-symmetric 4DEGB	Primary hair solutions (including regular black holes)	Conformal Kerr

$\phi = \cdots$	$G_{2,4}(X)$ (shift + parity sym. Horndeski)	$G_{2,3,4,5}(\phi, X)$ (Generic Horndeski)	$G_{2,4}(X)$ and $F_4(X)$ (shift + parity sym. beyond Horndeski)	DHOST
$\phi(r)$	BCL (Babichev-Charmousis-Lehébel)	4DEGB (Einstein-Gauss-Bonnet) + extensions without conformal invariance	Secondary hair solutions, homogeneous or not	Regular black holes (Kerr-Schild construction)
$qt+\psi(r)$ and $X =$ constant	Stealth Schwarzschild	$\emptyset$	$\emptyset$	Stealth Kerr and Disformed Kerr
$qt+\psi(r)$ and $X \neq$ constant	(New) Stealth Schwarzschild	Shift-symmetric 4DEGB	Primary hair solutions (including regular black holes)	Conformal Kerr

Stealth Schwarzschild

[E. Babichev, C. Charmousis, 2014]

$$S = \int \mathrm{d}^4x\sqrt{-g}\left\{R + \beta G^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi\right\}$$

Shift-symmetric Horndeski with a unique nonvanishing function $G_4 = 1 + \beta X$. Static, spherically-symmetric ansatz:

$$\mathrm{d}s^2 = -h(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2$$

No-hair argument [L. Hui, A. Nicolis, 2013], which requires in addition that $\phi = \phi(r)$. But the action depends on the scalar field only via its derivatives, so a **linear time dependence** is allowed:

$$\phi = qt + \psi(r)$$

$\rightsquigarrow$ Stealth Schwarzschild solution with **constant** $X = -(\partial\phi)^2/2$:

$$h(r) = f(r) = 1 - \frac{2M}{r}, \quad \psi'(r) = \pm q\frac{\sqrt{2Mr}}{r-2M}, \quad X = \frac{q^2}{2}$$

- Constant X allows to generalize this procedure and to find more general Horndeski [T. Kobayashi, N. Tanahashi, 2014] and DHOST theories [M. Minamitsuji, H. Motohashi, 2018] admitting such stealth Schwarzschild solutions
- Absence of the scalar field kinetic term in the action brings about perturbative problems (unlike stealth Schwarzschild-dS)

Recently, **new stealth Schwarzschild** [A. Bakopoulos, C. Charmousis, P. Kanti, N. L., T. Nakas, 2023] with $G_2 = 2\eta\sqrt{X}$ (non-standard kinetic term) and $G_4 = 1 + \lambda\sqrt{X}$, and **non-constant** X:

$$S = \int \mathrm{d}^4x\sqrt{-g}\left\{2\eta\sqrt{X} + \left(1 + \lambda\sqrt{X}\right)R + \frac{\lambda}{2\sqrt{X}}\left[(\Box\phi)^2 - (\phi_{\mu\nu})^2\right]\right\}$$

$$\phi = qt + \psi(r), \quad \psi'(r)^2 = \frac{q^2}{f^2(r)}\left[1 - \frac{\lambda f(r)}{\lambda + \eta r^2}\right], \quad X = \frac{\lambda q^2/2}{\lambda + \eta r^2}$$

Stealth Kerr with Hamilton-Jacobi scalar field

[C. Charmousis, M. Crisostomi, R. Gregory, N. Stergioulas, 2019]

Shift-symmetric quadratic DHOST with $c_{gw} = c$:

$$S = \int \mathrm{d}^4 x \sqrt{-g} \Big\{ f(X) R + P(X) + Q(X) \Box\phi + A_3(X) \phi^\mu \phi_{\mu\nu} \phi^\nu \Box\phi + \cdots \Big\}$$

If $X = (\partial\phi)^2 = X_0 = -q^2$, Ricci-flat metric $R_{\mu\nu} = 0$ implies

$$P(X_0) = P_X(X_0) = Q_X(X_0) = A_3(X_0) = 0$$

So such a theory admits a stealth Kerr solution. $X = -q^2$ can be linked to the mass-shell equation of a point-particle with mass q,

$$-q^2 = (\partial\phi)^2 = g^{\mu\nu} \partial_\mu\phi \, \partial_\nu\phi \equiv g^{\mu\nu} p_\mu p_\nu$$

i.e. ϕ is **Hamilton-Jacobi functional** of the geodesic: $\partial_\mu\phi = p_\mu$

Using Carter's integration of Kerr geodesics then gives

$$\phi = q\left(t \pm \int \frac{\sqrt{2Mr\left(r^2+a^2\right)}}{\Delta}\mathrm{d}r\right)$$

in the usual Boyer-Lindquist coordinates, with Kerr metric

$$\mathrm{d}s^2 = -\left(1-\frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2 - \frac{4Mar\sin^2\theta}{\Sigma}\mathrm{d}t\mathrm{d}\varphi + \frac{\sin^2\theta}{\Sigma}\left[\left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta\right]\mathrm{d}\varphi^2,$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 + a^2 - 2Mr$.

Again, absence of scalar field kinetic term $\rightsquigarrow$ scalar field perturbation equation is non-hyperbolic [C. de Rham, J. Zhang, 2019]

Disformed Kerr metric

[T. Anson, E. Babichev, C. Charmousis, M. Hassaine] [J. Ben Achour, H. Liu, H. Motohashi, S. Mukohyama, K. Noui]

Disformal transformation with constant factor D:

$$\widetilde{g}_{\mu\nu} = g^{\text{Kerr}}_{\mu\nu} - \frac{D}{q^2}\partial_\mu\phi\,\partial_\nu\phi$$

Disformed Kerr has mass $\widetilde{M} = \frac{M}{1+D}$ and angular mom. $\tilde{J} = \frac{J}{\sqrt{1+D}}$:

$$\begin{aligned}
\mathrm{d}\widetilde{s}^2 = & -\left(1 - 2\widetilde{M}r/\Sigma\right)\mathrm{d}t^2 - \frac{4\sqrt{1+D}\,\widetilde{M}ar\sin^2\theta}{\Sigma}\mathrm{d}t\mathrm{d}\varphi + \Sigma\,\mathrm{d}\theta^2 \\
& + \frac{\sin^2\theta}{\Sigma}\left[\left(r^2+a^2\right)^2 - a^2\Delta\sin^2\theta\right]\mathrm{d}\varphi^2 - 2D\frac{\sqrt{2\widetilde{M}r\left(r^2+a^2\right)}}{\Delta}\mathrm{d}t\mathrm{d}r \\
& + \frac{\Sigma\Delta - 2D\left(1+D\right)\widetilde{M}r\left(r^2+a^2\right)}{\Delta^2}\mathrm{d}r^2
\end{aligned}$$

Non-circular: $(t,\varphi) \to (-t,-\varphi)$ not a symmetry because of $\mathrm{d}t\mathrm{d}r$
$\rightsquigarrow$ Richer causal structure: static limit (ergosphere) + stationary limit (Killing horizon) + event horizon

Conformal Kerr metric

[E. Babichev, C. Charmousis and N. L.]

Conformal transformation of stealth Kerr metric gives a non-stationary black hole with FLRW behaviour when $r \to \infty$:

$$\widetilde{g}_{\mu\nu} = C(\phi)\, g^{\text{Kerr}}_{\mu\nu}, \quad \phi = q\tau \text{ (conformal time)},$$
$$C(\phi) = a^2(\phi/q) = a^2(\tau) \text{ (conformal FLRW scale factor)}$$

E.g. for vanishing rotation:

$$\mathrm{d}\widetilde{s}^2 = a^2(\tau)\left\{-\left(1-\frac{2M}{r}\right)\mathrm{d}\tau^2 + 2\sqrt{\frac{2M}{r}}\mathrm{d}\tau\mathrm{d}r + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\right\}$$

$$\phi = q\tau$$

$\phi = \cdots$	$G_{2,4}(X)$ (shift + parity sym. Horndeski)	$G_{2,3,4,5}(\phi, X)$ (Generic Horndeski)	$G_{2,4}(X)$ and $F_4(X)$ (shift + parity sym. beyond Horndeski)	DHOST
$\phi(r)$	BCL (Babichev-Charmousis-Lehébel)	4DEGB (Einstein-Gauss-Bonnet) + extensions without conformal invariance	Secondary hair solutions, homogeneous or not	Regular black holes (Kerr-Schild construction)
$qt + \psi(r)$ and $X =$ constant	Stealth Schwarzschild	$\emptyset$	$\emptyset$	Stealth Kerr and Disformed Kerr
$qt + \psi(r)$ and $X \neq$ constant	(New) Stealth Schwarzschild	Shift-symmetric 4DEGB	Primary hair solutions (including regular black holes)	Conformal Kerr

Framework

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big\{G_2(X) + G_4(X)R + G_{4X}\left[(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right] + F_4(X)\,\epsilon^{\mu\nu\rho\sigma}\epsilon^{\alpha\beta\gamma}{}_{\sigma}\phi_\mu\phi_\alpha\phi_{\nu\beta}\phi_{\rho\gamma}\Big\}$$

Shift-symmetry $\Rightarrow$ **Noether current** $J^\mu = -\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta(\partial_\mu\phi)}$, $\nabla_\mu J^\mu = 0$

Ansatz: static, spherically-symmetric metric, compatible scalar:

$$\mathrm{d}s^2 = -h(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2, \quad \phi = qt + \psi(r)$$

- $q \neq 0 \Rightarrow J^r \propto \mathcal{E}_{tr} \Rightarrow J^r = 0$ [E. Babichev, C. Charmousis, M. Hassaine, 2015]
- $q = 0 \Rightarrow J^r = \frac{C}{r^2\sqrt{h/f}} \Rightarrow J_\mu J^\mu = \frac{C^2}{r^4 h}$, so $J_\mu J^\mu < \infty \Rightarrow J^r = 0$

$\rightsquigarrow$ Focus on solutions with $J^r = 0$ and regular kinetic term X

Pure Horndeski black hole

[E. Babichev, C. Charmousis, A. Lehébel, 2017]

Idea: inspect carefully the form of J^r, in order to identify which Horndeski functionals enable a non-trivial scalar field from the equation $J^r = 0$ *for spherical symmetry*

$\rightsquigarrow$ Two couplings η and β, canonical kinetic term:

$$G_2 = \eta X, \quad G_4 = 1 + \beta\sqrt{-X}, \quad F_4 = 0$$

Solution: homogeneous metric ($h = f$) and static scalar field $\phi = \phi(r)$:

$$f(r) = 1 - \frac{2M}{r} - \frac{\beta^2}{2\eta r^2}, \quad \phi'(r) = \pm\frac{\sqrt{2}\beta}{\eta r^2\sqrt{f(r)}}$$

The kinetic term is finite apart from the central singularity:

$$X = -\frac{\beta^2}{\eta^2 r^4}$$

Beyond Horndeski black holes

[A. Bakopoulos, C. Charmousis, P. Kanti, N. L., 2017]

Other black holes with $\phi = \phi(r)$ can be found only by including a beyond Horndeski F_4 term, and share the property that for $M = 0$, $\mathrm{d}s^2 \neq$ flat. For example, for a canonical kinetic term, one has:

$$G_2 = \frac{8\eta\beta^2}{\lambda^2} X, \quad G_4 = 1 + 4\eta\beta\left(\sqrt{-X} + \beta X\right), \quad F_4 = -\frac{\eta\beta^2}{X}$$

with three couplings η, β, λ. Homogeneous metric:

$$f(r) = 1 - \frac{2M}{r} + \eta \frac{\arctan(r/\lambda) - \pi/2}{r/\lambda} = 1 - \frac{2M}{r} - \frac{\eta\lambda^2}{r^2} + \mathcal{O}\left(\frac{1}{r^4}\right)$$

The kinetic term is well-defined including now at $r = 0$:

$$X = -\frac{\lambda^4}{4\beta^2 (r^2 + \lambda^2)^2}, \quad \phi'(r) = \pm\frac{1}{\beta\sqrt{2f(r)}\left(1 + (r/\lambda)^2\right)}$$

Near $r = 0$:

$$f(r) = 1 + \eta - \frac{2M + \pi\eta\lambda/2}{r} - \frac{\eta r^2}{3\lambda^2} + \mathcal{O}\left(r^4\right),$$

$\rightsquigarrow$ threshold mass $M_0 = -\pi\eta\lambda/4$. For $M > M_0$, unique horizon; for $M < M_0$, two horizons if M is not too small, and zero horizon for small masses; for $M = M_0$, zero or one horizon according to $\eta \gtrless -1$, and $f(r)$ does not diverge at $r = 0$ but the spacetime remains singular there because $f(0) \neq 1$

Once a homogeneous solution is obtained, procedure to generalize it to non-homogeneous ones. For example:

$$G_2 = \frac{8\eta\beta^2}{\lambda^2}\frac{X}{1-\xi^2 X}, \quad G_4 = \frac{1 + 4\eta\beta\left(\sqrt{-X} + \beta X\right)}{1 - \xi^2 X},$$

$$F_4 = \frac{\xi^6 X^2 - \left(3\xi^2 + 4\eta\beta^2\right)\xi^2 X - 8\eta\beta\xi^2\sqrt{-X} - 4\eta\beta^2}{4X\left(1 - \xi^2 X\right)^2}$$

gives

$$f(r) = \frac{h(r)}{\left(1 - \xi^2 X\right)^2}$$

Primary hair solution

[A. Bakopoulos, C. Charmousis, P. Kanti, N. L., T. Nakas, 2023]

By including linear time dependence of the scalar, $\phi = qt + \psi(r)$, get primary hair solutions: the integration constant q, independent from the mass, appears in the metric. Theory functionals depend on two couplings λ (> 0) and η:

$$G_2 = -\frac{8\eta}{3\lambda^2}X^2, \quad G_4 = 1 - \frac{4\eta}{3}X^2, \quad F_4 = \eta,$$

$$h(r) = f(r) = 1 - \frac{2M}{r} + \eta q^4 \left(\frac{\pi/2 - \arctan(r/\lambda)}{r/\lambda} + \frac{1}{1 + (r/\lambda)^2} \right),$$

$$\psi'(r)^2 = \frac{q^2}{f^2(r)} \left[1 - \frac{f(r)}{1 + (r/\lambda)^2} \right], \quad X = \frac{q^2/2}{1 + (r/\lambda)^2}$$

$q = 0$: Schwarzschild, $q \neq 0$: departure from Schwarzschild

$$f(r) = 1 - \frac{2M}{r} + \eta q^4 \left(\frac{\pi/2 - \arctan(r/\lambda)}{r/\lambda} + \frac{1}{1 + (r/\lambda)^2} \right)$$

$$= 1 - \frac{2M}{r} + 2\lambda^2 \frac{\eta q^4}{r^2} + \mathcal{O}\left(\frac{1}{r^4}\right), \quad r \to \infty$$

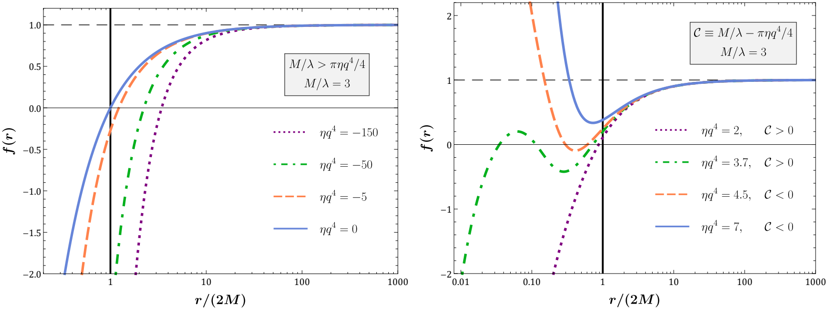

Figure 1: Left: $\eta < 0$, unique horizon greater than the Schwarzschild radius $r_S = 2M$. Right: $\eta > 0$, one, two, three or zero horizons, horizon smaller than Schwarzschild.

Regular spacetime (black hole or soliton)

For $M = \pi\eta q^4\lambda/4$, the central singularity disappears and all curvature invariants become infinitely regular:

$$f(r) = 1 - \frac{4M}{\pi\lambda}\left(\frac{\arctan(r/\lambda)}{r/\lambda} - \frac{1}{1+(r/\lambda)^2}\right)$$

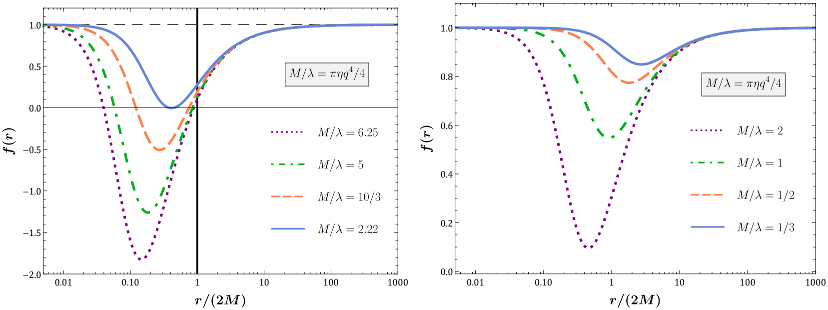

Figure 2: Left: Regular BH solutions. Right: regular solitonic solutions.

$\phi = \cdots$	$G_{2,4}(X)$ (shift + parity sym. Horndeski)	$G_{2,3,4,5}(\phi, X)$ (Generic Horndeski)	$G_{2,4}(X)$ and $F_4(X)$ (shift + parity sym. beyond Horndeski)	DHOST
$\phi(r)$	BCL (Babichev-Charmousis-Lehébel)	4DEGB (Einstein-Gauss-Bonnet) + extensions without conformal invariance	Secondary hair solutions, homogeneous or not	Regular black holes (Kerr-Schild construction)
$qt+\psi(r)$ and $X =$ constant	Stealth Schwarzschild	$\emptyset$	$\emptyset$	Stealth Kerr and Disformed Kerr
$qt+\psi(r)$ and $X \neq$ constant	(New) Stealth Schwarzschild	Shift-symmetric 4DEGB	Primary hair solutions (including regular black holes)	Conformal Kerr

The Gauss-Bonnet invariant $\mathcal{G}$

$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. According to Lovelock's theorem, $\mathcal{G}$ is the first higher-order correction to the Einstein-Hilbert action, giving the **Einstein-Gauss-Bonnet (EGB)** action:

$$S = \int \mathrm{d}^D x \sqrt{-g} \left\{ R + \hat{\alpha}\mathcal{G} \right\}$$

Problem: for $D = 4$, $\mathcal{G}$ is only a boundary term. Still, $\mathcal{G}$ gives non-trivial contributions when non-minimally coupled to a scalar field ϕ, e.g.

$$\int \mathrm{d}^4 x \sqrt{-g} \left\{ R + f(\phi)\mathcal{G} + \cdots \right\}$$

$\mathcal{G}$ also physically motivated by string theory [B. Zwiebach, 1985]

$\rightsquigarrow$ **What is the natural generalization/continuation of EGB gravity in four dimensions?**

Kaluza-Klein (KK) dimensional reduction

[C. Charmousis, B. Gouteraux, E. Kiritsis, 2012]

Start from the D-dimensional EGB theory:

$$\int \mathrm{d}^D x \sqrt{-g_{(D)}} \Big\{ R_{(D)} + \hat{\alpha} \mathcal{G}_{(D)} \Big\}$$

Diagonal decomposition of the D-dimensional metric

$$\mathrm{d}l^2_{(D)} = \mathrm{d}s^2_{(4)} + \mathrm{e}^{-2\phi} \mathrm{d}\widetilde{s}^2_{(n)}$$

yields (for short $n_k = n - k$):

$$\int \mathrm{d}^D x \sqrt{-g_{(D)}} R_{(D)} \propto \int \mathrm{d}^4 x \sqrt{-g}\, \mathrm{e}^{-n\phi} \Big\{ R + \widetilde{R} \mathrm{e}^{2\phi} + n\,(n-1)\,(\partial\phi)^2 \Big\}$$

and

$$\int \mathrm{d}^D x \sqrt{-g_{(D)}} \mathcal{G}_{(D)} \propto \int \mathrm{d}^4 x \sqrt{-g}\, \mathrm{e}^{-n\phi} \Big\{ \mathcal{G} + \widetilde{\mathcal{G}} \mathrm{e}^{4\phi} + 2\widetilde{R} \mathrm{e}^{2\phi} \Big[R + n_2 n_3\,(\partial\phi)^2 \Big] - 4 n n_1\, G^{\mu\nu} \phi_\mu \phi_\nu + 2 n n_1 n_2 \Box\phi\,(\partial\phi)^2 - n n_1^2 n_2\,(\partial\phi)^4 \Big\}$$

Singular limit

[H. Lu, Y. Pang, 2020]

The prescription $n \to 0$, $\hat{\alpha} \to \infty$, $n\hat{\alpha} = \text{const.} \equiv \alpha$ yields only finite terms, provided the following regularized curvature invariants are well-defined:

$$\widetilde{R}_{\text{reg}} = \lim_{n\to 0} \frac{\widetilde{R}}{n}, \quad \widetilde{\mathcal{G}}_{\text{reg}} = \lim_{n\to 0} \frac{\widetilde{\mathcal{G}}}{n},$$

and the term $\mathrm{e}^{-n\phi}\hat{\alpha}\mathcal{G}$ is dealt with as follows:

$$\mathrm{e}^{-n\phi}\hat{\alpha}\mathcal{G} = \underbrace{\cancel{\hat{\alpha}\mathcal{G}}}_{\text{BT}} - n\hat{\alpha}\phi\mathcal{G} + \mathcal{O}\left(n^2\right) \underset{n\to 0}{\longrightarrow} -\alpha\phi\mathcal{G}.$$

$\rightsquigarrow$ regularized Kaluza-Klein action or **4DEGB theory**:

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big\{R + \alpha\Big[-\phi\mathcal{G} + \widetilde{\mathcal{G}}_{\text{reg}}\mathrm{e}^{4\phi} + 2\widetilde{R}_{\text{reg}}\mathrm{e}^{2\phi}\Big(R + 6\left(\partial\phi\right)^2\Big) + 4G^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + 4\Box\phi\left(\partial\phi\right)^2 + 2\left(\partial\phi\right)^4\Big]\Big\}$$

4DEGB from generalized conformal invariance

[P. G. S. Fernandes, 2021]

Most general 4D action with second order field equations, and such that **the scalar field equation of motion has conformal symmetry**, i.e. is invariant under $g_{\mu\nu} \to \mathrm{e}^{2\sigma} g_{\mu\nu}$, $\phi \to \phi - \sigma$:

$$S = \int \mathrm{d}^4 x \sqrt{-g} \Big\{ R - 2\lambda \mathrm{e}^{4\phi} - \beta \mathrm{e}^{2\phi} \left[R + 6 \left(\partial\phi\right)^2 \right] + \alpha \left[-\phi \mathcal{G} + 4 G^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi + 4 \Box\phi \left(\partial\phi\right)^2 + 2 \left(\partial\phi\right)^4 \right] \Big\}$$

Same action as the regularized KK action with:

$$2\lambda = -\alpha \widetilde{\mathcal{G}}_{\text{reg}}, \quad \beta = -2\alpha \widetilde{R}_{\text{reg}}$$

Due to the *generalized conformal invariance*, geometric constraint:

$$2 g^{\mu\nu} \frac{\delta S}{\delta g^{\mu\nu}} + \frac{\delta S}{\delta \phi} \propto R + \frac{\alpha}{2} \mathcal{G}.$$

The $\beta \mathrm{e}^{2\phi} X$ term can be transformed into a canonical kinetic term for the scalar field $\Phi \equiv \mathrm{e}^{\phi}$

Flat internal space [R. A. Hennigar, D. Kubizňák, R. B. Mann, C. Pollack, 2020] [C. Charmousis, A. Lehébel, E. Smyrniotis and N. Stergioulas, 2021] (i.e. $\lambda = \beta = 0$):

$$f = 1 + \frac{r^2}{2\alpha}\left(1 \pm \sqrt{1 + \frac{8\alpha M}{r^3}}\right) = 1 - \frac{2M}{r} + \frac{4\alpha M^2}{r^4} + \mathcal{O}\left(\frac{1}{r^7}\right),$$

$$\phi = qt + \int \frac{\pm\sqrt{q^2 r^2 + f(r)} - f(r)}{r\, f(r)}\mathrm{d}r.$$ Horizons: $r_\pm = M \pm \sqrt{M^2 - \alpha}$

Maximally-symmetric internal space [H. Lu, Y. Pang, 2020] (i.e. $\lambda = \frac{3\beta^2}{4\alpha}$):

$$f = \text{idem}, \quad \phi(r) = \log\left(\frac{\sqrt{-2\alpha/\beta}}{r}\right) - \log\left(\cosh\left(c \pm \int \frac{\mathrm{d}r}{r\sqrt{f(r)}}\right)\right)$$

Internal space product of 2-spheres [P. G. S. Fernandes, 2021] (i.e. $\lambda = \frac{\beta^2}{4\alpha}$):

$$f = 1 + \frac{r^2}{2\alpha}\left(1 - \sqrt{1 + \frac{8\alpha M}{r^3} + \frac{8\alpha^2}{r^4}}\right) = 1 - \frac{2M}{r} - \frac{2\alpha}{r^2} + \mathcal{O}\left(\frac{1}{r^4}\right),$$

$$\phi(r) = \log\left(\sqrt{-2\alpha/\beta}/r\right).$$ Horizon: $r_+ = M + \sqrt{M^2 + \alpha}$

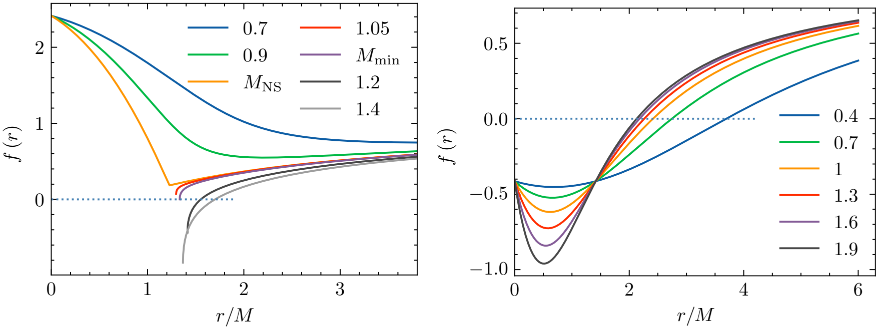

Figure 3: Metric function $f(r)$ for different values of $M/\sqrt{|\alpha|}$ for negative α (left plot) and positive α (right plot) for the 4DEGB solution in the case of internal space product of two-spheres. One has $f(0) = 1 + \sqrt{2}$ for $\alpha < 0$ and $f(0) = 1 - \sqrt{2}$ for $\alpha > 0$

Extension of the log case without conformal invariance

[E. Babichev, C. Charmousis, M. Hassaine, N. L., 2023]

$$S=\int \mathrm{d}^4x\sqrt{-g}\Big\{R-2\lambda_4 \mathrm{e}^{4\phi}-2\lambda_5\mathrm{e}^{5\phi}-\beta_4\mathrm{e}^{2\phi}\left(R+6\left(\partial\phi\right)^2\right)$$
$$-\beta_5\mathrm{e}^{3\phi}\left(R+12\left(\partial\phi\right)^2\right)-\alpha_4\left(\phi\mathcal{G}-4G^{\mu\nu}\phi_\mu\phi_\nu-4\Box\phi\left(\partial\phi\right)^2-2\left(\partial\phi\right)^4\right)$$
$$-\alpha_5\mathrm{e}^{\phi}\left(\mathcal{G}-8G^{\mu\nu}\phi_\mu\phi_\nu-12\Box\phi\left(\partial\phi\right)^2-12\left(\partial\phi\right)^4\right)\Big\},$$

The couplings with λ_5, β_5 and α_5 would correspond to a conformally-invariant scalar field in **five dims.** [J. Oliva and S. Ray, 2012]

First solution:

$$\lambda_4=\frac{\beta_4^2}{4\alpha_4},\quad \lambda_5=\frac{9\beta_5^2}{20\alpha_5},\quad \frac{\beta_5}{\beta_4}=\frac{2\alpha_5}{3\alpha_4},$$

$$f=1+\frac{2\alpha_5\eta}{3\alpha_4 r}+\frac{r^2}{2\alpha_4}\left[1-\sqrt{\left(1+\frac{4\alpha_5\eta}{3r^3}\right)^2+4\alpha_4\left(\frac{2M}{r^3}+\frac{2\alpha_4}{r^4}+\frac{8\alpha_5\eta}{5r^5}\right)}\right],$$

$$\phi=\ln\left(\eta/r\right),\quad \eta\equiv\sqrt{-2\alpha_4/\beta_4}$$

Second solution: $\lambda_4 = \beta_4 = \alpha_4 = 0, \quad \lambda_5 = 9\beta_5^2 / (20\alpha_5),$

$$f(r) = \frac{1}{1 + \frac{4\alpha_5 \eta}{3r^3}} \left[1 - \frac{2M}{r} - \frac{4\alpha_5 \eta}{15 r^3}\right], \quad \phi = \ln\left(\frac{\eta}{r}\right), \quad \eta \equiv 2\sqrt{\frac{-\alpha_5}{3\beta_5}}$$

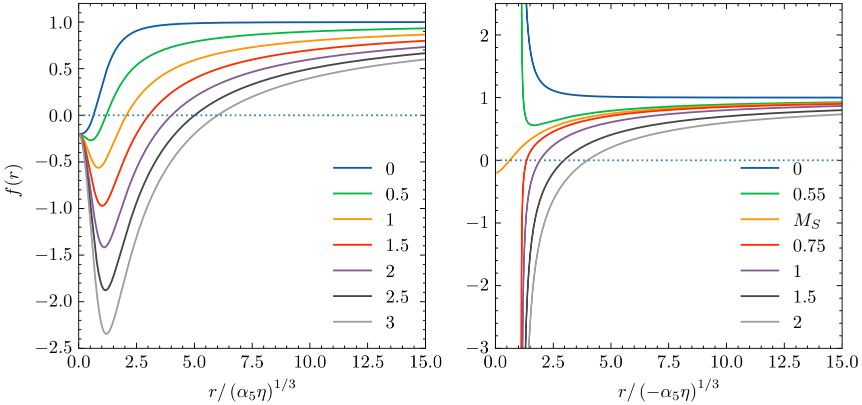

Figure 4: Metric profile $f(r)$ for $\alpha_5 > 0$ (left plot) or $\alpha_5 < 0$ (right plot) and different values of $M/|\alpha_5\eta|^{1/3}$

A rotating solution to 4DEGB?

[P. G. S. Fernandes, 2023]

Generic rotating Kerr-Schild ansatz:

$$\begin{aligned} \mathrm{d}s^2 = & -\left(1 - 2r\mathcal{M}\left(r,\theta\right)/\Sigma\right)\left(\mathrm{d}v - a\sin^2\theta\,\mathrm{d}\varphi\right)^2 \\ & + 2\left(\mathrm{d}v - a\sin^2\theta\,\mathrm{d}\varphi\right)\left(\mathrm{d}r - a\sin^2\theta\,\mathrm{d}\varphi\right) + \Sigma\,\mathrm{d}\Omega^2 \end{aligned}$$

$R + \frac{\alpha}{2}\mathcal{G} = 0$ gives $\mathcal{M}\left(r,\theta\right)$ in terms of arbitrary $M\left(\theta\right)$ and $q\left(\theta\right)$:

$$\mathcal{M}\left(r,\theta\right) = \frac{2\left(M - \frac{q}{2r}\right)}{1 \pm \sqrt{1 - \frac{8\alpha r}{\Sigma^3}\left(r^2 - 3a^2\cos^2\theta\right)\left(M - \frac{q}{2r}\right)}}$$

Yields a full solution for $\lambda = 0$ and constant scalar field $\phi = -\log\left(\beta\right)/2$, because then all equations become an identity but the geometric constraint $R + \frac{\alpha}{2}\mathcal{G} = 0$

$\rightsquigarrow$ also work for non-constant scalar field?

Conclusions: Stealth solutions and disformal transformations

$\phi = \cdots$	$G_{2,4}(X)$ (shift + parity sym. Horndeski)	$G_{2,3,4,5}(\phi, X)$ (Generic Horndeski)	$G_{2,4}(X)$ and $F_4(X)$ (shift + parity sym. beyond Horndeski)	DHOST
$\phi(r)$	BCL (Babichev-Charmousis-Lehébel)	4DEGB (Einstein-Gauss-Bonnet) + extensions without conformal invariance	Secondary hair solutions, homogeneous or not	Regular black holes (Kerr-Schild construction)
$qt+\psi(r)$ and $X =$ constant	Stealth Schwarzschild	$\emptyset$	$\emptyset$	Stealth Kerr and Disformed Kerr
$qt+\psi(r)$ and $X \neq$ constant	(New) Stealth Schwarzschild	Shift-symmetric 4DEGB	Primary hair solutions (including regular black holes)	Conformal Kerr

Conclusions: Non-stealth in parity and shift-sym. theories

$\phi = \cdots$	$G_{2,4}(X)$ (shift + parity sym. Horndeski)	$G_{2,3,4,5}(\phi, X)$ (Generic Horndeski)	$G_{2,4}(X)$ and $F_4(X)$ (shift + parity sym. beyond Horndeski)	DHOST
$\phi(r)$	BCL (Babichev-Charmousis-Lehébel)	4DEGB (Einstein-Gauss-Bonnet) + extensions without conformal invariance	Secondary hair solutions, homogeneous or not	Regular black holes (Kerr-Schild construction)
$qt+\psi(r)$ and $X =$ constant	Stealth Schwarzschild	$\emptyset$	$\emptyset$	Stealth Kerr and Disformed Kerr
$qt+\psi(r)$ and $X \neq$ constant	(New) Stealth Schwarzschild	Shift-symmetric 4DEGB	Primary hair solutions (including regular black holes)	Conformal Kerr

Conclusions: Non-stealth BHs in generic Horndeski

$\phi = \cdots$	$G_{2,4}(X)$ (shift + parity sym. Horndeski)	$G_{2,3,4,5}(\phi, X)$ (Generic Horndeski)	$G_{2,4}(X)$ and $F_4(X)$ (shift + parity sym. beyond Horndeski)	DHOST
$\phi(r)$	BCL (Babichev-Charmousis-Lehébel)	4DEGB (Einstein-Gauss-Bonnet) + extensions without conformal invariance	Secondary hair solutions, homogeneous or not	Regular black holes (Kerr-Schild construction)
$qt + \psi(r)$ and $X =$ constant	Stealth Schwarzschild	$\emptyset$	$\emptyset$	Stealth Kerr and Disformed Kerr
$qt + \psi(r)$ and $X \neq$ constant	(New) Stealth Schwarzschild	Shift-symmetric 4DEGB	Primary hair solutions (including regular black holes)	Conformal Kerr

Thank you for your attention!

$\phi = \cdots$	$G_{2,4}(X)$ (shift + parity sym. Horndeski)	$G_{2,3,4,5}(\phi, X)$ (Generic Horndeski)	$G_{2,4}(X)$ and $F_4(X)$ (shift + parity sym. beyond Horndeski)	DHOST
$\phi(r)$	BCL (Babichev-Charmousis-Lehébel)	4DEGB (Einstein-Gauss-Bonnet) + extensions without conformal invariance	Secondary hair solutions, homogeneous or not	Regular black holes (Kerr-Schild construction)
$qt+\psi(r)$ and $X =$ constant	Stealth Schwarzschild	$\emptyset$	$\emptyset$	Stealth Kerr and Disformed Kerr
$qt+\psi(r)$ and $X \neq$ constant	(New) Stealth Schwarzschild	Shift-symmetric 4DEGB	Primary hair solutions (including regular black holes)	Conformal Kerr