Degenerate Einstein-Maxwell theories

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I/ Introduction : Ghost-free U(1) vector-tensor interactions

- What are the most general interactions (in 4D) between gravity and electromagnetism

$$I\left[g_{\mu\nu}, A_\mu\right] = \int d^4x \sqrt{-g}\, \mathcal{L}\left[g_{\mu\nu}, \nabla_\alpha, R_{\mu\nu\sigma}{}^{\rho}, F_{\rho\sigma}\right], \quad (1)$$

with the same degrees of freedom as the Einstein-Maxwell theory, i.e. massless photons and gravitons ?

- ☐ Usually, higher order field equations generate new unstable degrees of freedom : Ostrogradski ghosts ;
- ☐ Unique U(1) vector-tensor theories admitting second order field equations, Horndeski (1976) :

$$I_{\text{2sd}}\left[g_{\mu\nu}, F_{\mu\nu}\right] = \int d^4x \sqrt{-g}\left(R - \mathcal{F}\left(F_{\mu\nu}F^{\mu\nu}, {}^*F_{\mu\nu}F^{\mu\nu}\right) + \gamma\, {}^*F_{\sigma\rho}{}^*F^{\mu\nu}R^{\sigma\rho}_{\mu\nu}\right), \quad (2)$$

where ${}^*F^{\rho\sigma} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}$ and $R^{\mu\nu}_{\sigma\rho} = g^{\gamma\mu}R_{\sigma\rho\gamma}{}^{\nu}$. Admits many exact non-singular solutions !

  - For $\gamma = 0$, yields Non-Linear Electrodynamics : Born-Infeld, Conformal Electrodynamics Bandos et al (2020) ;
  - For $\mathcal{F}(x, y) = \Lambda + \alpha x + \mu y^2$ Colléaux (2023) : Geodesically complete magnetic black string, strongly coupled non-singular Bianchi IX cosmology, non-singular electric $z = 2$ Lifshitz geometry.

- ☐ Can we go further and consider $I\left[R_{\mu\nu\sigma}{}^{\rho}, F_{\mu\nu}, \nabla_\sigma F_{\mu\nu}\right]$?

I/ Framework for answering : A detour on DHOST

- In the case of scalar-tensor theories, $\nabla_\alpha \nabla_\beta \phi$ already participates to Horndeski second order theories :

$$I_{\text{Horndeski}} = \int d^4x \sqrt{-g} \sum_{n=2}^{5} \mathcal{L}_n^{\text{H}} \qquad X = -\partial_\mu \phi \partial^\mu \phi \tag{3}$$

$$\begin{aligned} \mathcal{L}_2^{\text{H}} &:= G_2(\phi, X) \ , \quad \mathcal{L}_3^{\text{H}} := G_3(\phi, X) \Box \phi \ , \quad \mathcal{L}_4^{\text{H}} := G_4(\phi, X) R + G_{4,X} \delta^{\alpha\beta}_{\mu\nu} \nabla^\mu \nabla_\alpha \phi \nabla^\nu \nabla_\beta \phi \\ \mathcal{L}_5^{\text{H}} &:= G_5(\phi, X) G^\mu_\nu \nabla^\nu \nabla_\mu \phi - \frac{1}{6} G_{5,X} \delta^{\alpha\beta\rho}_{\mu\nu\sigma} \nabla^\mu \nabla_\alpha \phi \nabla^\nu \nabla_\beta \phi \nabla^\sigma \nabla_\rho \phi \, . \end{aligned} \tag{4}$$

- However, degenerate higher order scalar-tensor theories (DHOST), propagating only a massless graviton and a scalar field exist :

$$I_{\text{quadratic}} = \int d^4x \sqrt{-g} \left(f(\phi, X) R + C^{\mu\nu\rho\sigma}(g, \phi, \partial_\alpha \phi) \nabla_\mu \nabla_\nu \phi \nabla_\rho \nabla_\sigma \phi \right) \tag{5}$$

where C is the most general tensor built from $\partial_\sigma \phi$, providing that some degeneracy conditions (DC) on f and C hold ;

- ☐ Roughly speaking, DC are satisfied if a factorization occurs after an ADM decomposition,

$$\mathcal{L}_{\text{kin}} = \mathcal{K}^{ijkl} K_{ij} K_{kl} + \mathcal{B}^{ij} \ddot{\phi} K_{ij} + \mathcal{A} \ddot{\phi}^2 + V(\dot{\phi}) = \mathcal{K}^{ijkl} \left(K_{ij} + \mathcal{E}_{ij} \ddot{\phi} \right) \left(K_{kl} + \mathcal{E}_{kl} \ddot{\phi} \right) + V(\dot{\phi}) \tag{6}$$

where $\{\mathcal{K}, \mathcal{B}, \mathcal{A}, \mathcal{E}\}$ come from $\{f, C\}$ and depend on $\phi, \dot{\phi}, \partial_i \phi$.

I/ Degenerate Einstein-Maxwell theories

- The generalization of quadratic DHOST for U(1) gauge symmetry is

$$I_{\text{quadratic}} = \int d^4x\sqrt{-g}\left(\mathcal{F}\left(F^2, {}^*FF\right) + \frac{1}{4}\mathscr{A}^{\mu\nu\rho\sigma}\left(F_{\alpha\beta}, g_{\gamma\delta}\right)R_{\mu\nu\rho\sigma} + \mathscr{B}^{\gamma\mu\nu,\delta\rho\sigma}\left(F_{\alpha\beta}, g_{\gamma\delta}\right)\nabla_\gamma F_{\mu\nu}\nabla_\delta F_{\rho\sigma}\right),$$

where $\mathscr{A}$ and $\mathscr{B}$ are the most general tensors built from $F_{\rho\sigma}$ with the corresponding symmetries. Thus, need to :

- ☐ Classify $\mathscr{A}$ and $\mathscr{B}$;
- ☐ Perform an ADM decomposition of (7) and find the degeneracy conditions so that only massless photons and gravitons propagate ;
- ☐ Remark : The DC are partial differential equations between the free functions inside $\mathscr{A}$ and $\mathscr{B}$. Thus, to avoid spurious solutions corresponding to generic boundary terms or 4D redundancies, need minimal basis for $\mathscr{A}$ and $\mathscr{B}$;

- Proof of principle by U(1)-preserving disformal transformations Naruko, De Felice (2021) :

$$g_{\mu\nu} \longrightarrow \alpha\left(F^2, {}^*FF\right)g_{\mu\nu} + \beta\left(F^2, {}^*FF\right)F_\mu{}^\sigma F_{\sigma\nu} \tag{8}$$

If applied to $I_{2\text{sd}}\left[g_{\mu\nu}, F_{\mu\nu}\right]$ and the transformation is invertible, the degrees of freedom are preserved ;

I/ Further motivations : Bopp-Podosky ED and effective QED actions

- Bopp-Podolsky electrodynamics

$$I_{BP} = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{a^2}{2} \partial_\mu F^{\mu\nu} \partial^\rho F_{\rho\nu} \right) \quad (9)$$

  - ☐ Propagator (in Feynmann-t'Hooft gauge) :

$$P_{\mu\nu}(k) = \eta_{\mu\nu} \left(\frac{1}{k^2} - \frac{1}{k^2 - 1/a^2} \right) \quad (10)$$

  - ☐ So one photon and one ghost-like "Pauli-Villars" massive photon (a Proca field) of mass $1/a^2$: 5 degrees of freedom
  - ☐ Non-singular modified Coulomb potential for specific boundary cdts : $V(r) = \frac{q}{r} \left(1 - e^{-r/a}\right)$
- QED one-loop effective actions
  - ☐ Non minimal coupling "RF^2" and BP derivative term $\nabla F \nabla F$ in one (electron) loop corrections Drummond et al (1979);
  - ☐ Euler-Heisenberg non-perturbative one-loop effective action for QED in constant background EM field

$$L_{\text{eff}} = \mathcal{F}\left(F_{\mu\nu} F^{\mu\nu}, {}^*F_{\mu\nu} F^{\mu\nu}\right) \quad (11)$$

  - ☐ Non-perturbative one-loop effective action for (spinor) QED in slowly varying EM field Gusynin et al (1999)

I/ Derivative Expansion of the Effective Action for spinor QED in 3+1

$$A_{(j)\mu\nu} = \frac{-\bar{f}_j^2 \eta_{\mu\nu} + f_j F_{\mu\nu} + F^2_{\mu\nu} - i\bar{f}_j \overset{*}{F}_{\mu\nu}}{2(f_j^2 - \bar{f}_j^2)}, \qquad \begin{matrix} f_{1,2} = \pm i K_-, & f_{3,4} = \pm K_+; \\ \bar{f}_{1,2} = \mp K_+, & \bar{f}_{3,4} = \mp i K_-. \end{matrix} \quad K_+ = \sqrt{\sqrt{\mathcal{F}^2 + \mathcal{G}^2} + \mathcal{F}}, \qquad K_- = \sqrt{\sqrt{\mathcal{F}^2 + \mathcal{G}^2} - \mathcal{F}}.$$

$$\mathcal{F} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \qquad \mathcal{G} = \frac{1}{8} \epsilon^{\mu\nu\lambda\kappa} F_{\lambda\kappa} F_{\mu\nu}.$$

$$\begin{aligned}
& tr\langle x|U(\tau)|x\rangle = tr\langle x|U(\tau)|x\rangle_0 \\
& \times \Bigg[1 - \frac{i}{8} e F_{\nu\lambda,\mu\kappa} \sum_{j,l} \left(C^V(f_j, f_l) \left(A^{\nu\lambda}_{(j)} A^{\mu\kappa}_{(l)} + 2 A^{\nu\mu}_{(j)} A^{\lambda\kappa}_{(l)} \right) + 2 C^W(f_j, f_l) A^{\lambda\nu}_{(j)} A^{\mu\kappa}_{(l)} \right) \\
& - \frac{i}{18} e^2 F_{\nu\lambda,\mu} F_{\sigma\kappa,\rho} \sum_{j,l,k} \Bigg(9 C_1^{WW}(f_j, f_l, f_k) A^{\kappa\sigma}_{(j)} A^{\lambda\nu}_{(l)} A^{\mu\rho}_{(k)} + 9 C_2^{WW}(f_j, f_l, f_k) A^{\kappa\lambda}_{(j)} A^{\sigma\nu}_{(l)} A^{\mu\rho}_{(k)} \\
& + 6 C_1^{VW}(f_j, f_l, f_k) A^{\sigma\kappa}_{(j)} \left(A^{\nu\lambda}_{(l)} A^{\mu\rho}_{(k)} + A^{\nu\mu}_{(l)} A^{\lambda\rho}_{(k)} \right) + 6 C_2^{VW}(f_j, f_l, f_k) A^{\sigma\kappa}_{(j)} A^{\nu\rho}_{(l)} A^{\lambda\mu}_{(k)} \\
& - C_1^{VV}(f_j, f_l, f_k) \left(A^{\nu\lambda}_{(j)} A^{\kappa\sigma}_{(l)} A^{\mu\rho}_{(k)} + A^{\nu\mu}_{(j)} A^{\kappa\rho}_{(l)} A^{\lambda\sigma}_{(k)} + 2 A^{\nu\lambda}_{(j)} A^{\kappa\rho}_{(l)} A^{\mu\sigma}_{(k)} \right) \\
& - C_2^{VV}(f_j, f_l, f_k) \left(A^{\nu\sigma}_{(j)} A^{\kappa\lambda}_{(l)} A^{\mu\rho}_{(k)} + A^{\nu\rho}_{(j)} A^{\kappa\mu}_{(l)} A^{\lambda\sigma}_{(k)} + 2 A^{\nu\sigma}_{(j)} A^{\kappa\mu}_{(l)} A^{\lambda\rho}_{(k)} \right) \\
& - 2 C_3^{VV}(f_j, f_l, f_k) \left(A^{\nu\lambda}_{(j)} A^{\kappa\mu}_{(l)} A^{\sigma\rho}_{(k)} + A^{\kappa\rho}_{(j)} A^{\nu\sigma}_{(l)} A^{\lambda\mu}_{(k)} \right) - C_4^{VV}(f_j, f_l, f_k) A^{\nu\kappa}_{(j)} A^{\lambda\mu}_{(l)} A^{\sigma\rho}_{(k)} \\
& - C_5^{VV}(f_j, f_l, f_k) A^{\nu\kappa}_{(j)} \left(A^{\lambda\sigma}_{(l)} A^{\mu\rho}_{(k)} + A^{\lambda\rho}_{(l)} A^{\mu\sigma}_{(k)} \right) \Bigg) \Bigg],
\end{aligned}$$

II/ Classification of quadratic interactions

- $\mathscr{A}^{\mu\nu\rho\sigma}$ and $\mathscr{B}^{\gamma\mu\nu,\delta\rho\sigma}$ are constructed from (Cayley-Hamilton theorem in 4D) :
  - ☐ 2 functions $\{F^2, {}^*FF\}$ or equivalently $\{F^2, F^4\}$;
  - ☐ 2 symmetric matrices $\{g_{\mu\nu}, F_\mu{}^\alpha F_{\alpha\nu}\}$, and 2 antisymmetric ones $\{F_{\mu\nu}, F_\mu{}^\alpha F_{\alpha\beta} F^\beta{}_\nu\}$;
  - ☐ Using Levi-Civita tensor is redundant ;
- Define

$$F_I^{\mu\nu} \equiv \{g^{\mu\nu}, F^{\mu\nu}, F^{\mu\alpha} F_\alpha{}^\nu, F^{\mu\alpha} F_{\alpha\beta} F^{\beta\nu}\} \tag{12}$$

- Classification of $\mathscr{B}^{\gamma\mu\nu,\delta\rho\sigma}(F_I ... F_K) \nabla_\gamma F_{\mu\nu} \nabla_\delta F_{\rho\sigma}$
  - ☐ Fully dressed vertices : 2 vectors $u^\mu_{(1,2)IJ} \equiv (F_I F_J \nabla F)^\mu_{1,2}$, 1 tensor $t^{\mu\nu\sigma}_{IJK} \equiv F_I^{\mu\bar{\mu}} F_J^{\nu\bar{\nu}} F_K^{\sigma\bar{\sigma}} \nabla_{\bar{\mu}} F_{\bar{\nu}\bar{\sigma}}$;
  - ☐ Thus 5 family of scalars : $\{u_1 u_1, u_2 u_2, u_1 u_2, tt_1, tt_2\}$ and 124 of them in total ;

$$\mathscr{B}^{\gamma\mu\nu,\delta\rho\sigma}(F_{\alpha\beta}, g_{\gamma\delta}) \nabla_\gamma F_{\mu\nu} \nabla_\delta F_{\rho\sigma} = \sum_{i=1}^{124} \alpha_i S_i \tag{13}$$

  - ☐ Need to use Bianchi identity $\nabla_{[\mu} F_{\nu\sigma]} = 0$: 70
  - ☐ Need to use DDIs (dimensionally dependant identities) : 88 for instance

$$\begin{aligned} \mathscr{D}_{IJK,AB} &\equiv F_{Ib_1}^{[a_1} F_{Jb_2}^{a_2} F_{Kb_3}^{a_3} F_{Aa_4}^{a_4} F_{Ba_5}^{a_5]} (\nabla F \nabla F)^{b_1 b_2 b_3}_{[a_1 a_2 a_3]} = 0 \\ \mathscr{D}_{IJK,ABC} &\equiv F_{Ii}^{j} F_{Jb_1}^{[a_1} F_{Kb_2}^{a_2} F_{Aa_3}^{a_3} F_{Ba_4}^{a_4} F_{Ca_5}^{a_5]} (\nabla F \nabla F)^{b_1 b_2 i}_{[a_1 a_2] j} = 0 \end{aligned} \tag{14}$$

II/ Classification of quadratic interactions

- Considering boundary terms as well (complicated) we found that $\mathscr{A}$ and $\mathscr{B}$ can be reduced to :

$$\mathscr{A}^{\mu\nu\rho\sigma} = \beta_1 g^{\mu\rho} g^{\nu\sigma} + \beta_2 F^{\mu\nu} F^{\rho\sigma} + \beta_3 F^{\mu\rho}_{(2)} g^{\nu\sigma} + \beta_4 F^{\mu\rho}_{(2)} F^{\nu\sigma}_{(2)}, \tag{15}$$

and

$$\begin{aligned}\mathscr{B}^{\gamma\mu\nu,\delta\rho\sigma} =& \alpha_1 g^{\gamma\sigma} g^{\nu\delta} g^{\rho\mu} + \alpha_2 F^{\rho\mu}_{(2)} g^{\gamma\nu} g^{\delta\sigma} + \alpha_3 F^{\delta\rho}_{(2)} g^{\gamma\nu} g^{\sigma\mu} + \alpha_4 F^{\delta\mu}_{(2)} g^{\gamma\sigma} g^{\nu\rho} + \alpha_5 F^{\rho\mu}_{(2)} g^{\gamma\sigma} g^{\nu\delta} \\ &+ \alpha_6 F^{\delta\mu} F^{\rho\sigma} g^{\gamma\nu} - \alpha_7 F^{\mu\nu} F^{\delta\rho} g^{\gamma\sigma} + \alpha_8 F^{\delta\rho}_{(2)} F^{\sigma\mu}_{(2)} g^{\gamma\nu} + \alpha_9 F^{\gamma\rho}_{(2)} F^{\mu\delta}_{(2)} g^{\nu\sigma} + \alpha_{10} F^{\nu\delta}_{(2)} F^{\mu\rho}_{(2)} g^{\gamma\sigma} \\ &+ \alpha_{11} F^{\gamma\rho} F^{\mu\nu} F^{\delta\sigma}_{(2)} - \alpha_{12} F^{\rho\mu} F^{\nu\delta}_{(3)} g^{\gamma\sigma} + \alpha_{13} F^{\gamma\sigma}_{(2)} F^{\nu\delta}_{(2)} F^{\rho\mu}_{(2)} + \alpha_{14} F^{\rho\sigma} F^{\delta\mu}_{(2)} g^{\gamma\nu} + \alpha_{15} F^{\delta\mu}_{(2)} F^{\gamma\rho}_{(3)} g^{\nu\sigma}.\end{aligned} \tag{16}$$

All the coefficients α, β, γ are functions of the two electromagnetic invariants available in four dimensions

III/ Degenerate theories : Lagrangian decompositions

- Recall the action

$$I\left[g_{\mu\nu}, F_{\mu\nu}\left(A_\mu\right)\right] = \int d^4x\sqrt{-g}\left(\frac{1}{4}\mathscr{A}^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} + \mathscr{B}^{\gamma\mu\nu,\delta\rho\sigma}\nabla_\gamma F_{\mu\nu}\nabla_\delta F_{\rho\sigma}\right), \tag{17}$$

- Similarly to DHOSTs, introduce equivalent action :

$$I_{\text{eq}}\left[g_{\mu\nu}, F_{\mu\nu}, \lambda_{\mu\nu}, A_\mu\right] = I\left[g_{\mu\nu}, F_{\mu\nu}\right] + \int d^4x\sqrt{-g}\lambda^{\mu\nu}\left(\partial_\mu A_\nu - \partial_\nu A_\mu - F_{\mu\nu}\right), \tag{18}$$

Due to U(1) gauge symmetry, there is no time derivative of the magnetic part of the two-form $F_{\mu\nu}$

$$\nabla_\sigma F_{\mu\nu} = \lambda^{\rho}_{\sigma\mu\nu}\dot{E}_\rho + \Lambda^{\alpha\beta}_{\sigma\mu\nu}K_{\alpha\beta} + (\dots)\, a + (\dots)\, DE + (\dots)\, DB \tag{19}$$

where D is the spatial covariant derivative, the time derivative of the electric field is encoded into the following spatial quantity

$$\dot{E}_\mu \equiv n^\sigma\nabla_\sigma E_\mu + n_\mu a^\sigma E_\sigma, \tag{20}$$

After IBP

$$\begin{aligned} I_{\text{kin}} &= \int d^4x\sqrt{-g}\left(\mathscr{C}^{\mu\nu\rho\sigma}K_{\mu\nu}K_{\rho\sigma} + 2\mathscr{D}^{\mu\nu\sigma}K_{\mu\nu}\dot{E}_\sigma + \mathscr{E}^{\rho\sigma}\dot{E}_\rho\dot{E}_\sigma\right), \\ I_{\text{lin}} &= \int d^4x\sqrt{-g}\left(\mathscr{F}^{\mu\nu}K_{\mu\nu} + \mathscr{G}^{\mu}\dot{E}_\mu\right), \end{aligned} \tag{21}$$

III/ Degenerate theories : Constraints and degrees of freedom

- Decomposition of $A_\mu = -A_0 n_\mu + \hat{A}_\mu$ and $\lambda_{\mu\nu} = \varepsilon_{\mu\nu\sigma}\lambda^\sigma - n_{[\mu}\pi_{\nu]}$

$$\int d^4x\sqrt{-g}\lambda^{\mu\nu}\left(\partial_\mu A_\nu - \partial_\nu A_\mu - F_{\mu\nu}\right) = \int d^4x\sqrt{-g}\left(\pi^\mu\partial_t\hat{A}_\mu + A_0 G + \lambda^\mu C_\mu - \pi_\mu E^\mu\right) \quad (22)$$

- Start from 30-dimensional non-physical phase-space with :
  - A_0 and λ_i (and N and N^i) are Lagrange multipliers ;
  - From $F_{\mu\nu} \longrightarrow \{E^i, B^i\}$ and their momenta $\{\pi_E^i, \pi_B^i\}$, From $A_\mu \longrightarrow \hat{A}_i$ and its momentum π^i (coming from $\lambda_{\mu\nu}$) ;
  - 5 first class constraints : H_i, H_0, G : vector, scalar and Gauss constraints ;
  - 6 second class ;

$$C_i = B_i - \epsilon_{ijk}\partial^j\hat{A}^k \ , \quad \pi_B^i \approx 0 \quad (23)$$

  - Therefore 7 degrees from freedom : 2 gravitons, 2 photons, 3 ghosts, if possible to invert

$$\begin{bmatrix}\pi_E^\alpha \\ \pi^{\mu\nu}\end{bmatrix} = \begin{bmatrix}\mathscr{E}^{\alpha\beta} & \mathscr{D}^{\rho\sigma\alpha} \\ \mathscr{D}^{\mu\nu\beta} & \mathscr{C}^{\mu\nu\rho\sigma}\end{bmatrix}\begin{bmatrix}\dot{E}_\beta \\ K_{\rho\sigma}\end{bmatrix} + \begin{bmatrix}\mathcal{G}^\alpha \\ \mathcal{F}^{\mu\nu}\end{bmatrix} \quad (24)$$

  - In order to avoid ghosts, we need both degeneracy and further constraints.

III/ Degenerate theories : Quasi-linear electrodynamics

- Consider flat Minskowski space in Cartesian coordinates so that

$$
\begin{aligned}
I_{\text{kin}} &= \int d^4x \mathscr{E}^{\rho\sigma}(E,B)\,\dot{E}_\rho \dot{E}_\sigma\,, \\
I_{\text{lin}} &= \int d^4x \left(\mathscr{G}_1^{\mu\alpha\beta}(E,B)\,\partial_\alpha E_\beta + \mathscr{G}_2^{\mu\alpha\beta}(E,B)\,\partial_\alpha B_\beta\right) \dot{E}_\mu\,,
\end{aligned}
\tag{25}
$$

- Quasi-linear degenerate theories obtained when no second derivatives $\mathscr{E}^{\alpha\beta}=0$. This condition yields five theories :

$$
\mathscr{L}_{\text{QL}} = \sum_{p=1}^{5} \gamma_p \mathscr{L}_p\,, \tag{26}
$$

where γ_p are U(1) functions and

$$
\begin{aligned}
&\mathscr{L}_1 = F_{(2)}^{\mu\nu}\nabla_{[\gamma|}F_\mu{}^\gamma \nabla_{|\lambda]}F_\nu{}^\lambda\,, \qquad \mathscr{L}_2 = F_{(2)}^{\mu\nu}\nabla_{[\mu|}F_{\nu\gamma}\nabla_{|\lambda]}F_\gamma{}^\lambda\,, \\
&\mathscr{L}_3 = F^{\mu\nu}\nabla_{[\mu}F^2\nabla_{\gamma]}F_\nu{}^\gamma\,, \qquad \mathscr{L}_4 = F_{(2)}^{\mu\nu}F_{(2)}^{\rho\sigma}\nabla_{[\nu|}F_{\mu\rho}\nabla_{|\gamma]}F_\rho{}^\gamma\,, \\
&\mathscr{L}_5 = \frac{P^2}{16}\nabla_\sigma F_{\mu\nu}\nabla^\mu F^{\nu\sigma} + \frac{F^2}{8}F^{\mu\nu}\nabla_\mu F_{\nu\gamma}\nabla^\gamma F^2 - 2F^{\mu\nu}F_{(3)}^{\rho\sigma}\nabla_\sigma F_{\nu\gamma}\nabla^\gamma F_{\mu\rho} + F_{(2)}^{\mu\nu}F_{(2)}^{\rho\sigma}\nabla_\nu F_{\mu\rho}\nabla_\gamma F_\rho{}^\gamma\,.
\end{aligned}
\tag{27}
$$

III/ Degenerate theories : Third order non-minimal coupling interaction

- If we require $\mathscr{E}=0$ and $\mathscr{D}=0$, so that only terms quadratic in the extrinsic curvature appear in the kinetic term, we obtain,

$$\mathscr{L}_{\mathsf{Grav}} = aR + b\,{}^*F_{\sigma\rho}\,{}^*F^{\mu\nu}R^{\sigma\rho}_{\mu\nu} + c\,\mathcal{L}\,, \tag{28}$$

where a, b, c are coupling constants. The new term is given by

$$\mathcal{L} = \frac{1}{4}\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}R^{\alpha\beta}_{\mu\nu}\left(F^{\gamma}_{2\rho}F^{\delta}_{2\sigma} + F^2F_{\rho\sigma}F^{\gamma\delta}\right) - 2F^2F_2^{\mu\nu}G_{\mu\nu} - 2\left(\mathscr{L}_1 - 2\mathscr{L}_2 + \mathscr{L}_3\right), \tag{29}$$

where the $\mathscr{L}_i$ are given by Eq(27).

  - ☐ Similar to the unique curvature invariant in 4D leading to third order equations of motion Lovelock (1969) ;

$$C = {}^*R^{\mu\nu}_{\rho\sigma}\,{}^*R^{\rho\sigma}_{\alpha\beta}\,{}^*R^{\alpha\beta}_{\mu\nu} \tag{30}$$

  - ☐ However, similarly to $R+\alpha C$ which propagates 2 gravitons plus 3 Ostrogradski ghosts, we expect $\mathcal{L}$ to be unstable (Hamiltonian linear in momentum thus unbounded from below) Crisostomi et al(2018) ;

IV/ Conclusion :

- We classified quadratic U(1)-vector-tensor theories in ∇F and U(1) non-minimal couplings linear in curvature ;
- Some degenerate theories have been found (in particular quasi-linear in flat space), but there are hints they still propagate Ostrogradski ghosts ;
- Can we obtain a classification of ghost-free theories like in DHOSTS ?
- If so, will the physically viable theories among them reduce to U(1)-preserving Disformal transformations of $I_{2\text{sd}}[A, g]$?
- Remark that U(1) Disformal transformations of Einstein-Maxwell theory have been studied in Gumrukcuoglu, Namba (2020), in Minamitsuji (2020) for disformed Kerr-Newman (breaks circularity condition), in Bittencourt (2023), where exotic singularities have been found in disformed Maxwell ED ;
- Use the tools to obtain minimal basis of invariants to obtain/reduce effective action in QFT (in which lots of redundancy enters) : possible interesting cancellations or computational efficiency or new truncations ? What about gravitational actions ?

Thank you for your attention !