Fully consistent rotating black holes in the cubic Galileon theory

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The model

Theory of gravity

$$S(\mathbf{g}, \phi) = \int \left[\xi (R - 2\Lambda) - \eta (\partial\phi)^2 + \gamma' (\partial\phi)^2 \Box\phi\right] \sqrt{-g} \mathrm{d}x^4.$$

Restriction to $\Lambda = 0$ and $\eta = 0$

$$S(\mathbf{g}, \phi) = \int \left[R + \gamma (\partial\phi)^2 \Box\phi\right] \sqrt{-g} \mathrm{d}x^4.$$

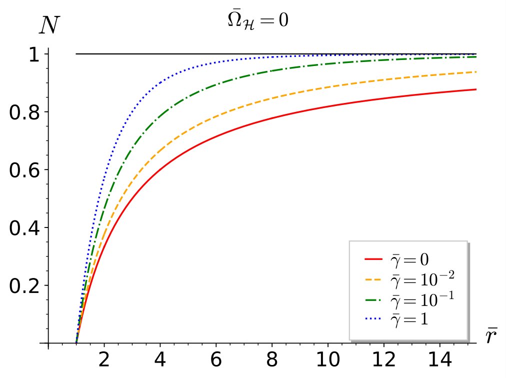
Field ansatz

$$\phi = qt + \Psi$$

- the action contains only derivatives of ϕ.
- the geometry is time-independent.
- q is a parameter of the solutions.

Black hole solutions

- The time-dependence in ϕ enables to evade the uniqueness theorem.
- Exact non-rotating black hole solutions can be found.
- Schwarzschild or isotropic coordinates.



Quasi-isotropic coordinates

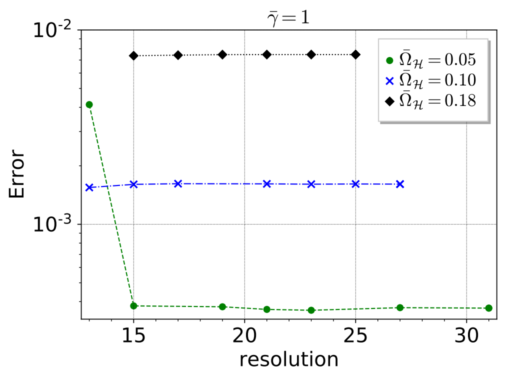
- Rotating solutions were constructed using quasi-isotropic coordinates:

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & 0 & 0 & g_{t\varphi} \\ 0 & A^2 & 0 & 0 \\ 0 & 0 & A^2 r^2 & 0 \\ g_{t\varphi} & 0 & 0 & B^2 r^2 \sin^2\theta \end{pmatrix}$$

- Only a subset of Einstein's equation components is solved.
- A valid choice iff the spacetime is circular (link to a generalized Papapetrou theorem).

Circularity condition is not verified.

The component (t, r) of Einstein's equation is not verified when rotation is included.



One needs to move away from quasi-circular coordinates and use more general ones.

3+1 formalism

Foliation of spacetime gives rise to the standard 3+1 metric :

$$\mathrm{d}s^2 = -\left(N^2 - B_i B^i\right)\mathrm{d}t^2 + 2B_i \mathrm{d}x^i \mathrm{d}t + \gamma_{ij}\mathrm{d}x^i \mathrm{d}x^j$$

The lapse N, the shift B^i and spatial metric γ_{ij} are the unknowns.

Definition : the second fundamental form is the extrinsic curvature tensor K_{ij} (first derivative of γ_{ij}).

The equations are the projections of Einstein's equations:

- the Hamiltonian constraint $H = 0$
- the momentum constraints $M^i = 0$
- the evolution equations $E_{ij} = 0$

Field contribution

- $T_{\mu\nu} = \gamma \left[\partial_{(\mu} \phi \partial_{\mu)} \partial\phi^2 - \Box\phi\partial_\mu\phi\partial_\nu\phi - \frac{1}{2} g_{\mu\nu} \partial^\rho\phi\partial_\rho\partial\phi^2 \right]$ contains only derivative of Ψ ($\phi = qt + \Psi$).
- The same is true for the 3+1 projections E, P^i and S_{ij}.
- Given axisymmetry, one can work with $\Psi_r = \partial_r \Psi$ and $\Psi_\theta = 1/r\partial_\theta \Psi$ so that
$$D_i\Psi = (\Psi_r, \Psi_\theta, 0) .$$
- One equation comes variation of the action wrt ϕ (see later...).
- The other one is $\partial_r (r\Psi_\theta) = \partial_\theta \Psi_r$.

Gauge choice

Stationarity $\implies \partial_t = 0$.

Einstein equations are not all independent, need to enforce gauge conditions (i.e. choice of coordinates).

- maximal slicing $K = 0$.
- spatial harmonic gauge $V^i = \gamma^{kl} \left(\Gamma^i_{kl} - \bar{\Gamma}^i_{kl} \right) = 0$.

Enforcing the gauge

In the 3+1 equations

- Remove all the occurrences of K and V^i in the equations.
- It gives a well-posed system of equations.
- Check, a posteriori, that $K = 0$ and $V^i = 0$.

With the spatial harmonic gauge, $R_{ij} = -\frac{1}{2}\gamma^{kl}\bar{D}_k\bar{D}_l\gamma_{ij} + \text{first order}$ (analogous of Lorenz gauge in gravitational waves).

Black hole as an apparent horizon

Definition

- One demands that it is a sphere.
- The normal to the sphere is $\tilde{s}$ (such that $\tilde{s}_i \tilde{s}^i = 1$).
- The expansion must vanish : $\Theta = D_i \tilde{s}^i + \tilde{s}^i \tilde{s}^j K_{ij} = 0$.

Additional conditions

- The horizon does not move : $\tilde{s}_i B^i = N$
- The horizon has no shear : $B^i_{\parallel} = \Omega \left(\partial_\varphi\right)^i$
- Ω encodes the rotation state of the black hole.
- It can be rewritten : $B^i = N\tilde{s}^i + \Omega \left(\partial_\varphi\right)^i$

Choice of time

- Consider a change of time coordinate $t' = t + \alpha\left(x^i\right)$.
- The condition $K = 0$ leads to a second order equation on α.
- The coordinate change is defined up to boundary conditions.
- This implies that one can choose freely the lapse on the horizon.
- A simple choice is $N = N_{\rm const}$ (for instance 0.5)

Choice of spatial coordinates

- Consider a change of time coordinate $x'^i = x^i + \xi\left(x^i\right)$.
- The condition $V^i = 0$ leads to a second order equation on ξ.
- The coordinate change is defined up to boundary conditions.
- It seems that it leads to three freely specifiable components.
- However the location of the horizon is fixed $\Longrightarrow$ one must have $\xi^r = 0$.
- This implies that one can choose freely two components of γ on the horizon.
- A simple choice $\gamma_{r\theta} = 0$ and $\gamma_{r\varphi} = 0$.

Degenerate equations

Example in 1D

- Consider the equation $a\left(r\right)f'' + b\left(r\right)f' + c\left(r\right) = 0$.
- $a\left(r\right) = 0$ at the inner boundary.
- One can not impose any boundary condition on f at the inner boundary.
- The equation is its own boundary condition : $b\left(r\right)f' + c\left(r\right) = 0$.

Application to the 3+1 system

- One looks at the terms involving ∂_r^2 for all the unknown fields, in all the equations.
- Some terms are multiplied by $\left(N^2 - B_i B^i\right)$ which vanishes on the horizon (see Eq. for B^i).
- A detailed analysis of the coupled system must be done.
- One can show that three equations are degenerate : the angular components of the evolution equation.
- $E_{\theta\theta}$, $E_{\theta\varphi}$ and $E_{\varphi\varphi}$ do not require boundary conditions.

Final complication

- The spherical part of the condition $\Theta = 0$ is automatically verified if the gauge conditions are fulfilled.
- It follows that the spherical part of another field can be freely specify.
- The Y_0^0 part of γ_{rr} is freely chosen.

Boundary conditions on the metric quantities

- $N = N_{\mathrm{const}}$ (time coordinate freedom).
- $\gamma_{r\theta} = 0$ and $\gamma_{r\varphi} = 0$ (spatial gauge freedom).
- $\gamma_{rr} = g_{\mathrm{const}}$ for $l = m = 0$ and $\Theta = 0$ otherwise.
- $B^i = N\tilde{s}^i + \Omega\left(\partial_\varphi\right)^i$ (horizon at fixed location, with no shear).
- $E_{\theta\theta} = 0$, $E_{\theta\varphi} = 0$ and $E_{\varphi\varphi} = 0$ (degenerate equations).

Field equation

- The field equation is a conservation equation $\nabla_\mu J^\mu = 0$ with

$$J_\mu = \gamma \left[\partial_\mu \phi + \frac{1}{2} \partial_\mu \left(\partial \phi \right)^2 \right].$$

- In terms of Ψ_r and Ψ_θ, the equation is of second order.
- It is degenerate on the horizon : only one boundary condition.

Naive choice

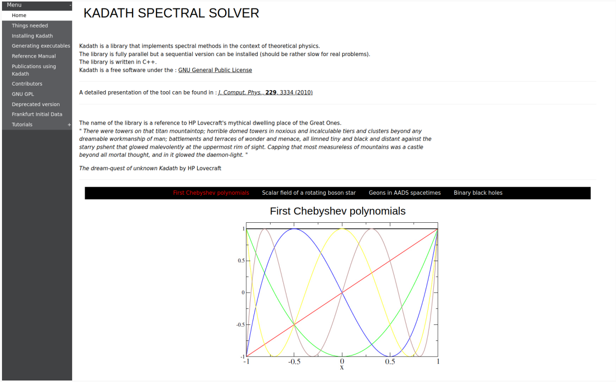
- Enforce $\Psi_r = 0$ at infinity.
- Leads to non-valid solutions (i.e. with $K \neq 0$ and $V^i \neq 0$).

Valid choice

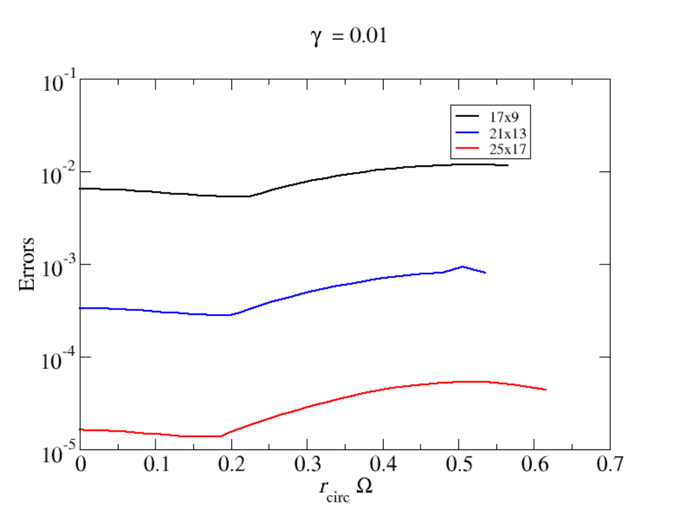
- Enforce $V^r = 0$ on the horizon.
- Different from other hairy black holes (different order ?).
- The compatibility conditions is solved with $\Psi_\theta = 0$ at infinity.

Numerics

- The equations are solved by means of the Kadath library *https://kadath.obspm.fr/*
- Enables the use of spectral methods in the context of GR and theoretical physics.
- Features : orthonormal spherical basis, multi-domain , compactification, Newton-Raphson iteration, parallel.

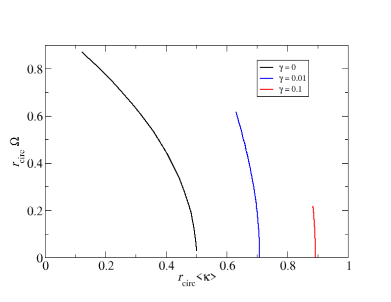


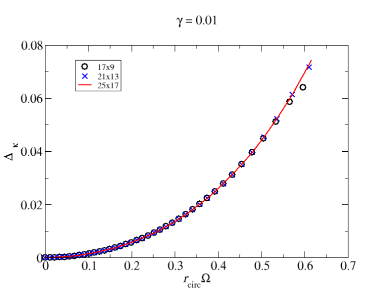
Convergence of the errors



First fully consistent numerical rotating black hole solutions in this theory.

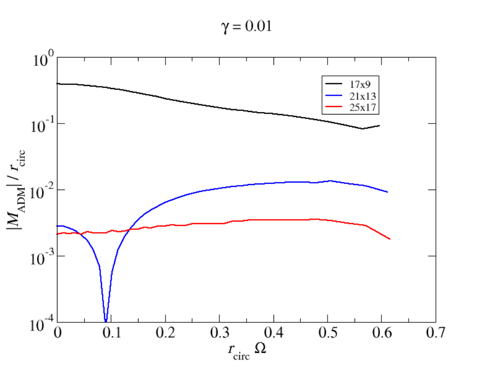
Zeroth law of BH thermodynamics





One can check that $E = T_{\mu\nu} n^\mu n^\nu < 0$ so that the weak energy condition does not hold. There is no reason why the solutions should obey the zeroth law.

Vanishing ADM mass



The dominant energy condition does not hold so the positive energy theorem does not apply.

Summary

- Application of a formalism with maximal slicing and spatial harmonic gauge.
- Apparent horizon in equilibrium.
- First fully consistent rotating solutions.
- Violation of the zeroth law of BH thermodynamics.

References

- First numerical solutions : K. Van Aelst *et al.*, Class. Quant. Grav., 37, 035007 (2020).
- Formalism : P. Grandclément *et al.*, Phys. Rev. D, 105, 104011 (2022).
- New solutions : P. Grandclément, arXiv:2308.11245.