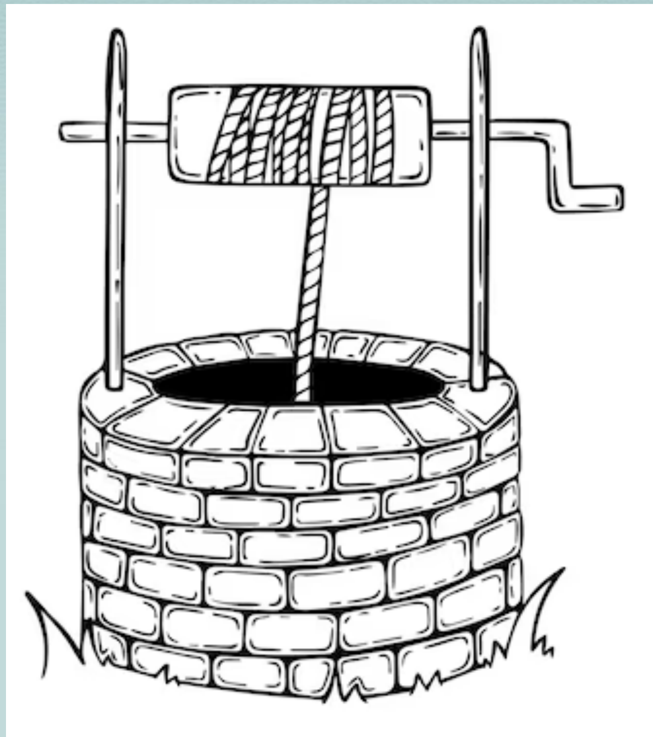
FROM BOSSUT TO RAYLEIGH-PLESSET: HOW CLASSICAL MECHANICS (INCLUDING HYDRODYNAMICS) CAN SHED LIGHT ON COSMOLOGY

"I possess the world all the better as I am more adept at miniaturizing it. »
Gaston Bachelard
(The poetics of Space)



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Featured in Physics

A Rapidly Expanding Bose-Einstein Condensate: An Expanding Universe in the Lab

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We study the dynamics of a supersonically expanding, ring-shaped Bose-Einstein condensate both experimentally and theoretically. The expansion redshifts long-wavelength excitations, as in an expanding universe. After expansion, energy in the radial mode leads to the production of bulk topological excitations—solitons and vortices—driving the production of a large number of azimuthal phonons and, at late times, causing stochastic persistent currents. These complex nonlinear dynamics, fueled by the energy stored coherently in one mode, are reminiscent of a type of "preheating" that may have taken place at the end of inflation.

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Subject Areas: Atomic and Molecular Physics, Cosmology, Quantum Physics

We understand the phonon's behavior during the expansion epoch in terms of a 1D equation for the phonon amplitude χ_m,

$$\frac{\partial^2 \chi_m}{\partial t^2} + \left[2\gamma_m(t) + \frac{\dot{R}}{R}\right]\frac{\partial \chi_m}{\partial t} + [\omega(t)]^2 \chi_m = 0$$

after expansion to extract $\omega_{m,i}/\omega_{m,f}$, shown in Fig. 2(c). At any given time, the phonon oscillation frequency is $\omega(t) = c_\theta(t)m/R(t)$, where $c_\theta(t)$ is the azimuthal speed of sound at time t. As the ring expands, both the atomic

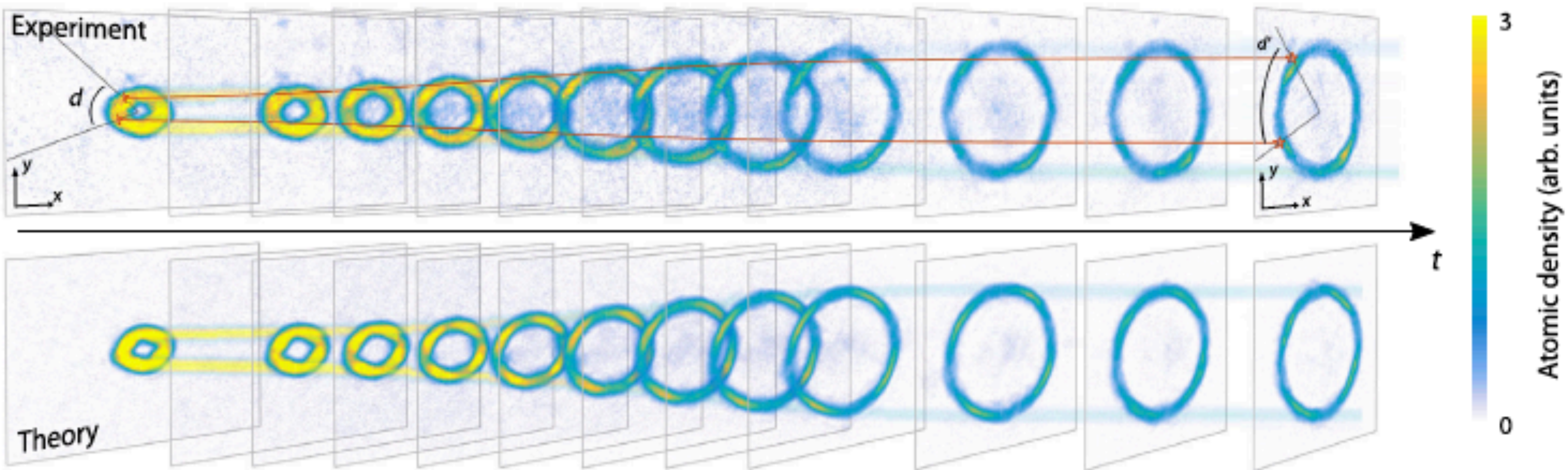


FIG. 1. Measured (top) and simulated (bottom) supersonic expansion of the ring with scale factor $a = R_f/R_i = 4.1(3)$, where $R_f = 46.4(1.4)$ μm [$R_i = 11.3(4)$ μm] is the final (initial) radius [44]. An initial distance d transforms into a larger distance d'. The time elapsed in the figure is approximately 15 ms.

Gravitational Waves in an Expanding Universe

- Perturbed FRWL metric (ignoring scalars and vectors):

$$ds^2 = -dt^2 + a^2(t)[(\delta_{ij} + h_{ij})dx^i dx^j]$$

$$|h_{ij}| \ll 1$$
$$h^i_i = \partial_j h^j{}_i = 0$$

- from Einstein equations

$$\ddot{h}_{ij}(\vec{x},t) + 3H\dot{h}_{ij}(\vec{x},t) - \frac{\nabla^2}{a^2}h_{ij}(\vec{x},t) = 16\pi G \Pi^{TT}_{ij}(\vec{x},t)$$

← source: tensor anisotropic stress

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}$$

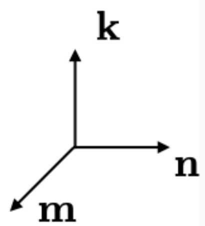
- Fourier transform, and polarisation components +, x Hubble radius $\frac{1}{H} = \frac{a}{\dot{a}}$ and comoving Hubble radius $\frac{1}{aH} = \frac{1}{\dot{a}}$

$$h_{ij}(\mathbf{x},t) = \sum_{r=+,\times} \int \frac{d^3\mathbf{k}}{(2\pi)^3} h_r(\mathbf{k},t)\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, e^r_{ij}(\hat{\mathbf{k}})$$

$$\Pi_{ij}(\mathbf{x},t) = \sum_{r=+,\times} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Pi_r(\mathbf{k},t)\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, e^r_{ij}(\hat{\mathbf{k}})$$

k= comoving wave number

$$e^+_{ij}(\hat{\mathbf{k}}) = \hat{m}_i \hat{m}_j - \hat{n}_i \hat{n}_j$$
$$e^\times_{ij}(\hat{\mathbf{k}}) = \hat{m}_i \hat{n}_j + \hat{n}_i \hat{m}_j$$
$$e^r_{ij}(\hat{\mathbf{k}})\, e^{r'}_{ij}(\hat{\mathbf{k}}) = 2\,\delta_{rr'}$$



- The equation decouples for each polarisation mode. In terms of conformal time $\eta = \int \frac{dt}{a(t)}$

$$h''_r(\mathbf{k},\eta) + 2\mathcal{H}\, h'_r(\mathbf{k},\eta) + k^2 h_r(\mathbf{k},\eta) = 16\pi G a^2 \Pi_r(\mathbf{k},\eta)$$

Note from Friedmann equation: $\mathcal{H}^2 = H^2 a^2 = \frac{8\pi G}{3} a^2 \bar{\rho}$

Courtesy Daniele Steer

https://indico.math.cnrs.fr/event/5761/contributions/4578/attachments/2769/3550/lecture2.pdf

Analogue Cosmology at Low Cost with a Varying Length Pendulum ?

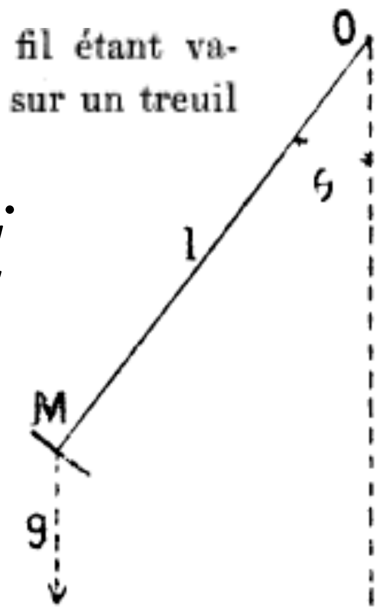
During this era, the expansion of the universe is sufficiently rapid to create particle, antiparticle pairs (Schrodinger, 1939, 1940; Parker, 1966; Sexl and Urbantke, 1969). Pair production by the gravitational field of the expanding universe is analogous to excitation of a pendulum caused by changing the length of its string. One can also picture the process as caused by gravitational tidal forces sufficiently strong to pull pairs out of the vacuum. Although such pair production is already of some significance, the density of particles *newly* created by the expansion remains small with respect to the density of particles already present, until one goes back to a somewhat earlier era, when the temperature was within a few orders of magnitude of the Planck temperature (10^{32} °K). An important feature of this particle creation is that it tends to rapidly damp out initial anisotropies in the expansion of the universe, so that by the time the temperature has fallen to 10^{30} °K the expansion is essentially isotropic (Zeldovich, 1970; Lukash and Starobinsky, 1974; Hu and Parker, 1978; Hartle and Hu, 1980). There is also an interesting connection between conformal invariance and particle creation which tends to suppress the creation of quarks, leptons, and photons if the expansion is isotropic (Parker, 1966, 1968, 1969, 1971). There is a possibility that this connection may leave its mark on the baryon-to-entropy ratio (Parker, 1981).

L. Parker, Particle creation in expanding universes, Physical Review Letters, 21 (8), p. 562 (1968).
L. Parker, Particle Creation by the Expansion of the Universe, Fundamentals of cosmic physics, 7, p. 201-239 (1982).

The Varying Length Pendulum

La vitesse du point M est $l\,\frac{d\theta}{dt}$, et son moment par rapport à l'axe passant en O est $l^2\,\frac{d\theta}{dt}$.

Fig. 1.



vue pratique, que, par le fait des oscillations, la tension du fil étant variable, la force nécessaire pour enrouler uniformément le fil sur un treuil ne saurait non plus demeurer constante.

$$T/m = g\cos\theta + l(\dot{\theta})^2 - \ddot{l}$$

Les forces extérieures se réduisent à la *gravité* g, appliquée en M, dont le moment est $gl\sin\theta$.

Quand θ augmente, cette force tend à diminuer l'angle. On aura donc, d'après le théorème rappelé,

$$\frac{d}{dt}\left(l^2\,\frac{d\theta}{dt}\right) + gl\sin\theta = 0.$$

On en déduira l'équation différentielle

$$\frac{d^2\theta}{dt^2} + \frac{2}{l}\,\frac{dl}{dt}\,\frac{d\theta}{dt} + \frac{g}{l}\sin\theta = 0. \quad (1)$$

$$\dot{\theta} = \frac{d\theta}{dt}$$

$$\ddot{\theta} = \frac{d^2\theta}{dt^2}$$

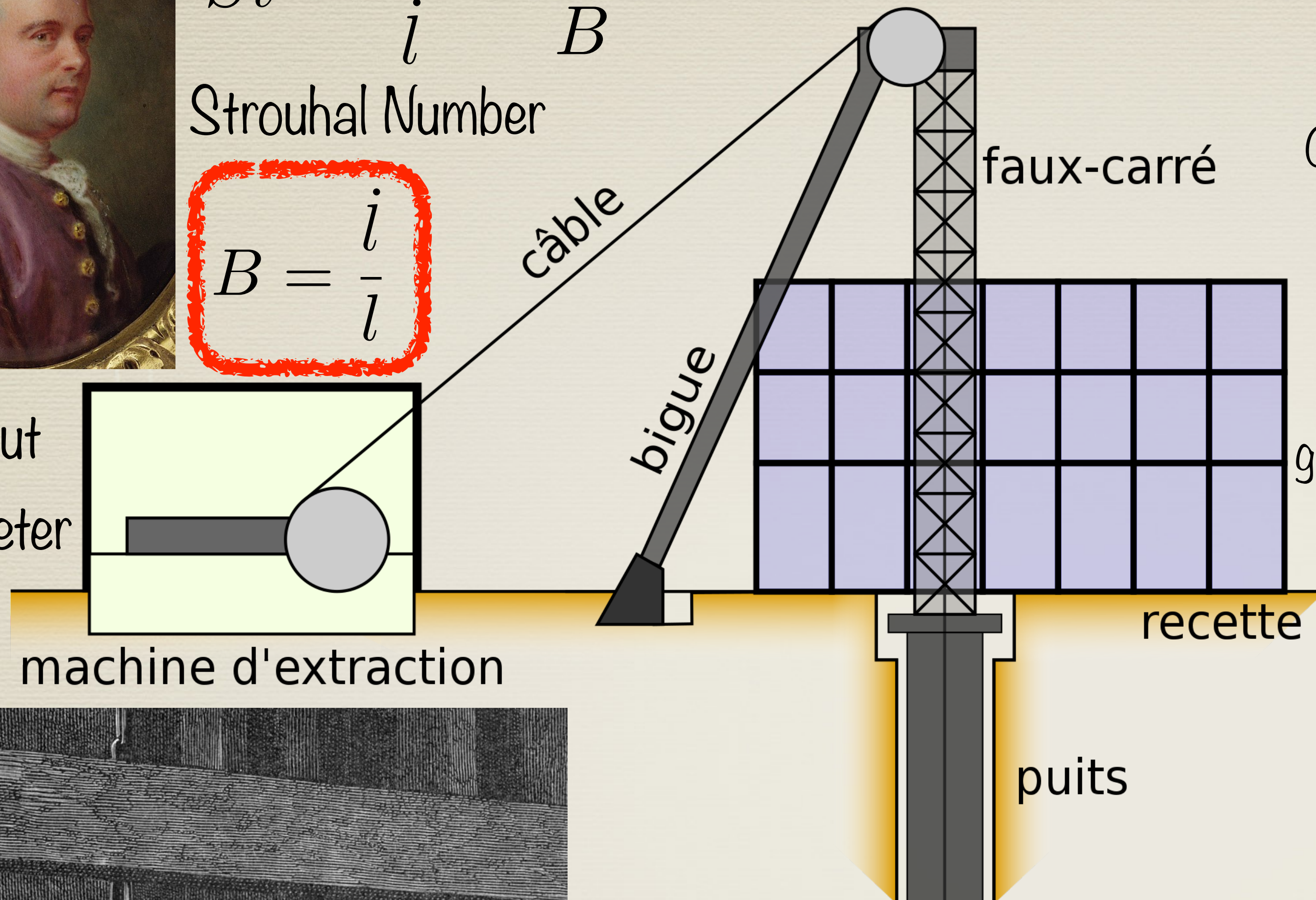
Charles Bossut's Varying Length Pendulum $\omega^2 = \frac{g}{l}$

$$St = \frac{\omega l}{\dot{l}} = \frac{\omega}{B}$$

Strouhal Number

$$B = \frac{\dot{l}}{l}$$

Bossut parameter

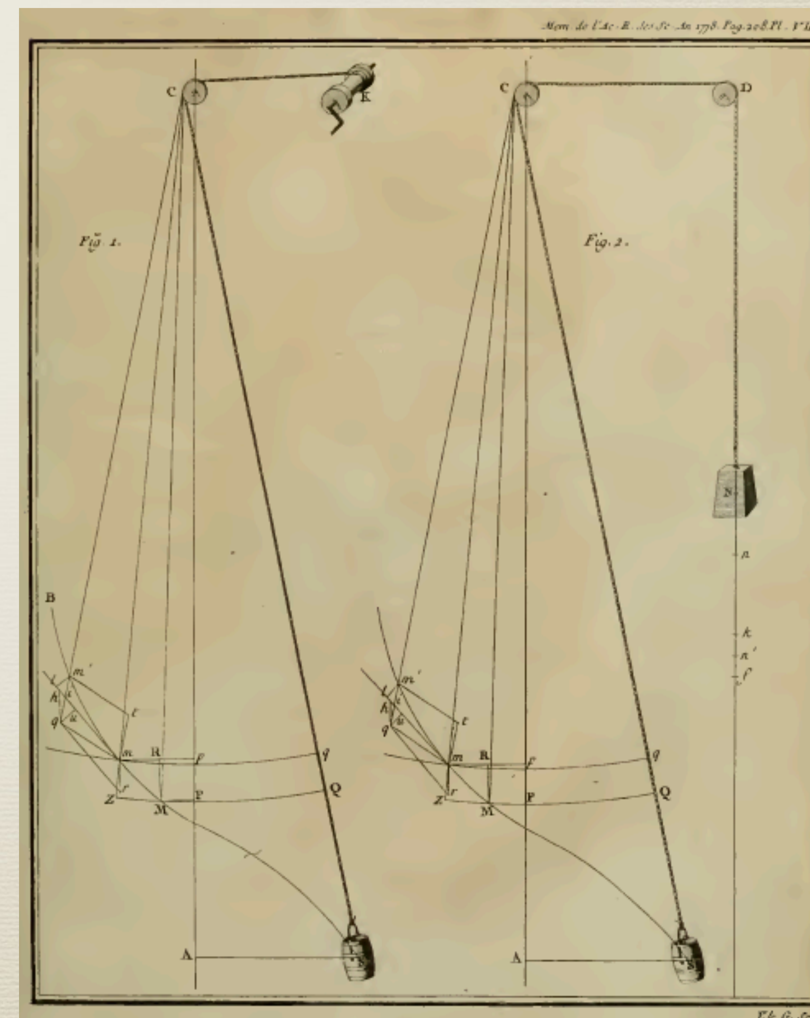


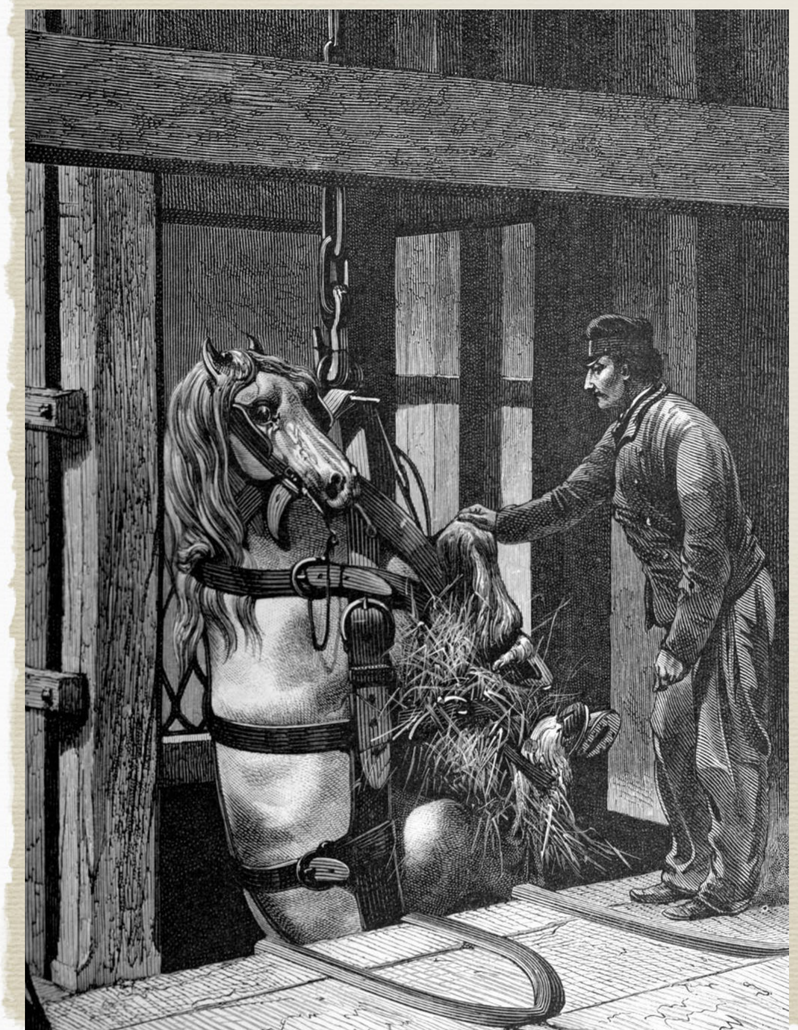
$$\ddot{\theta} + 2\frac{\dot{l}}{l}\dot{\theta} + \omega^2 \sin(\theta) = 0$$

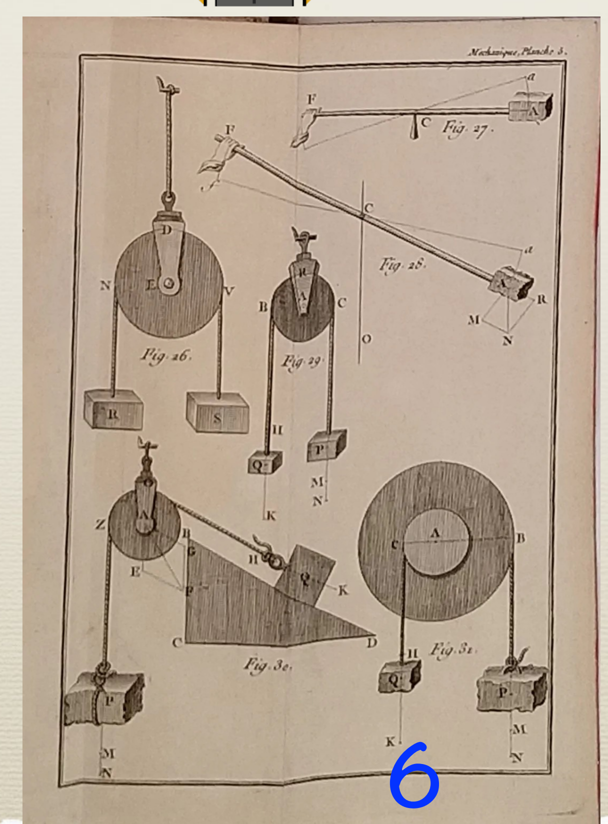
C. Bossut, Sur le mouvement d'un pendule dont la longueur est variable, Mémoires de l'Académie Royale des Sciences de Paris, lû le 5 Septembre 1778, p. 199-209 (1778).

J.-N. Haton de La Goupillière, Oscillations des bennes non guidées, Annales des mines, 15 (10), p. 531-577, Juin (1909).

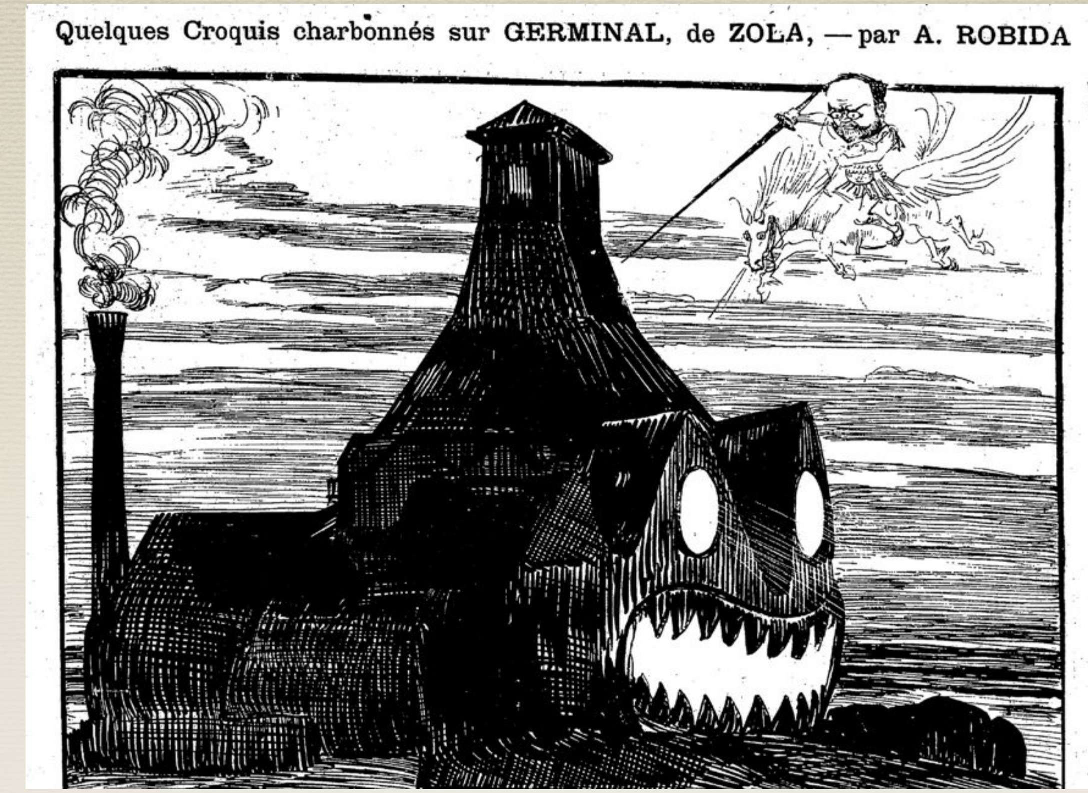
https://patrimoine.mines-paristech.fr/

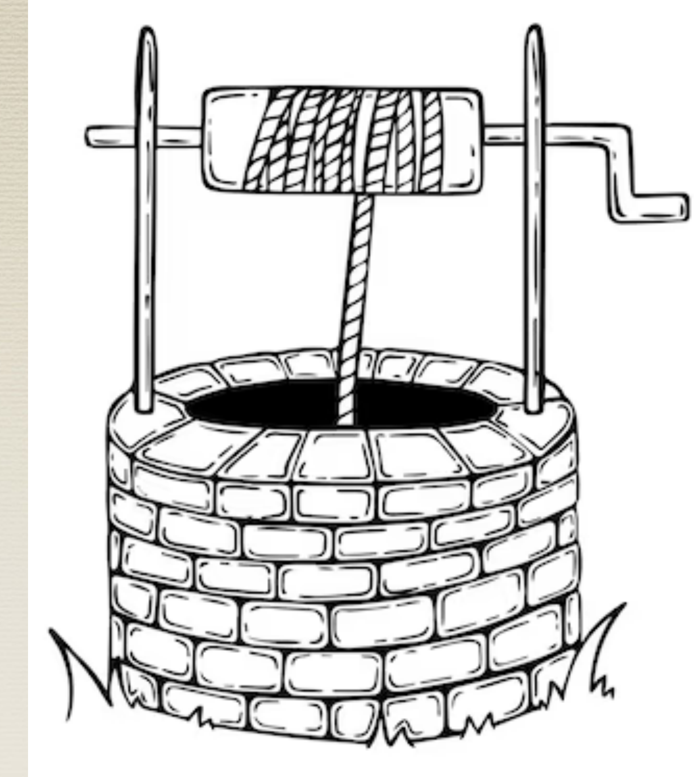






$$T/m = g\cos\theta + l(\dot{\theta})^2 - \ddot{l}$$





« This pit, packed at the bottom of a hollow... seemed to have the evil air of a greedy beast, crouching there to eat the world. And the Voreux, at the bottom of his hole, with his pile of wicked beasts, crushed himself more, breathed with bigger and longer breath, looking embarrassed by his painful digestion of human flesh. »

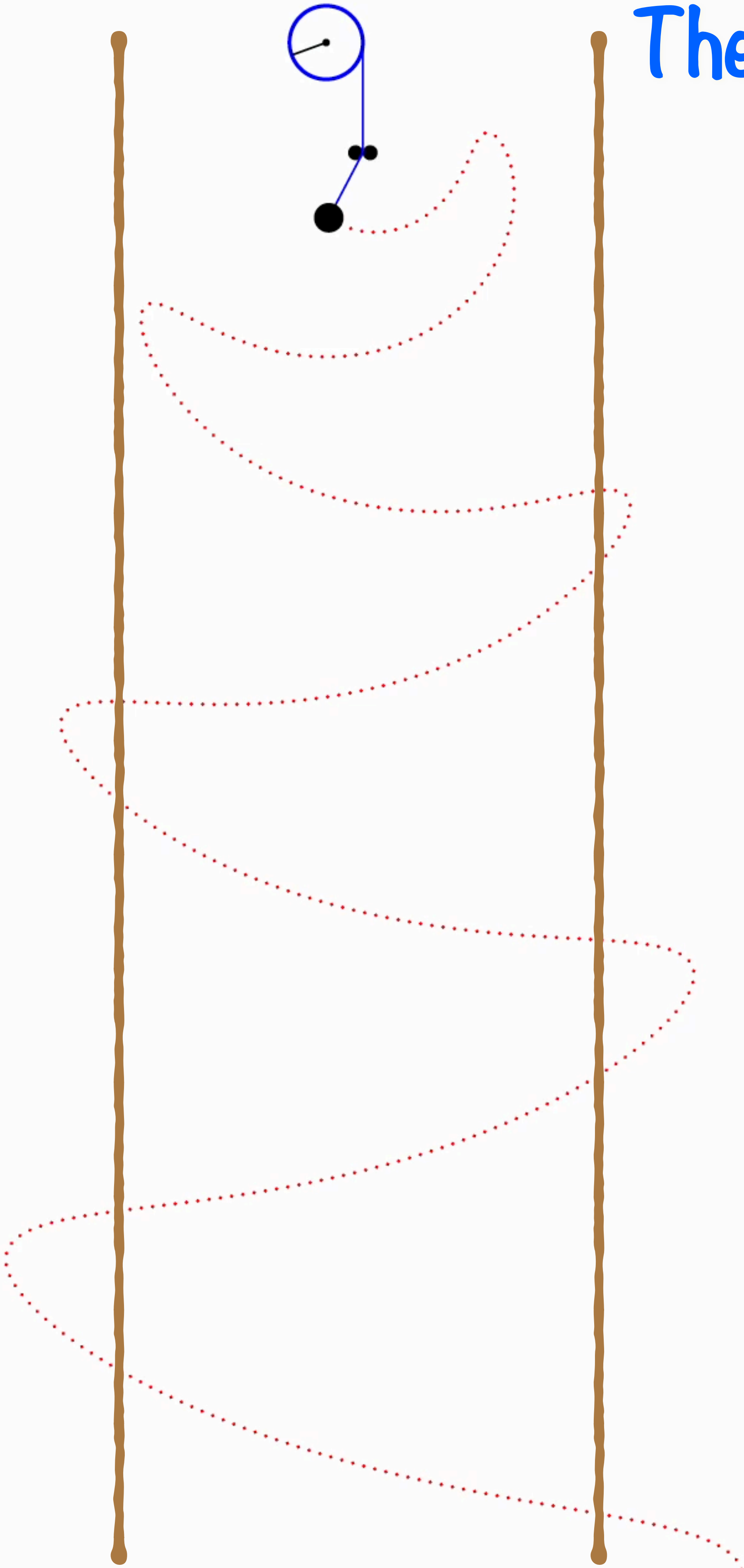
Emile Zola, Germinal (1885).

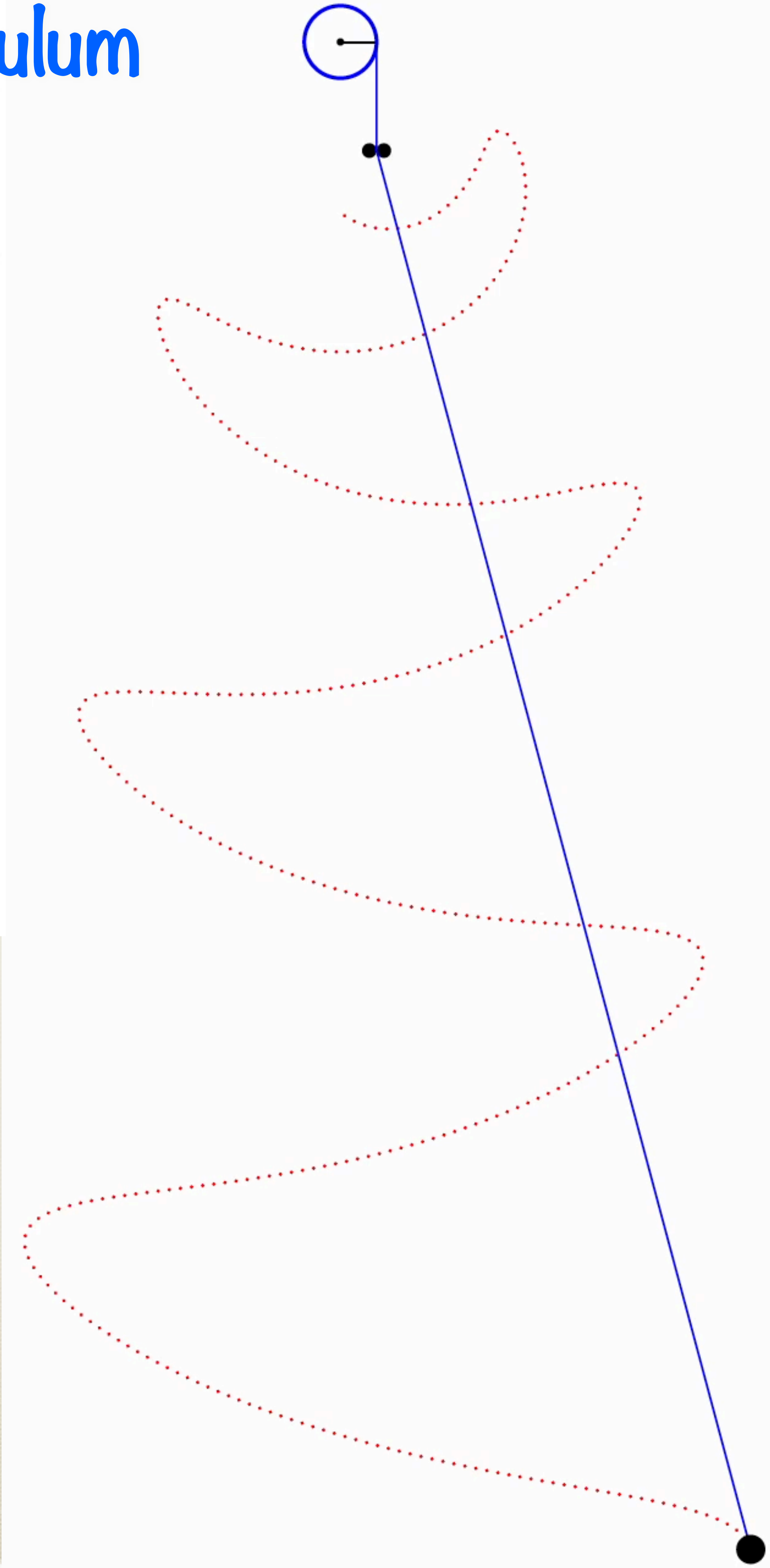
Cuffat : Wooden tub, suspended from a cable and used in mine shafts to transport material and personnel.

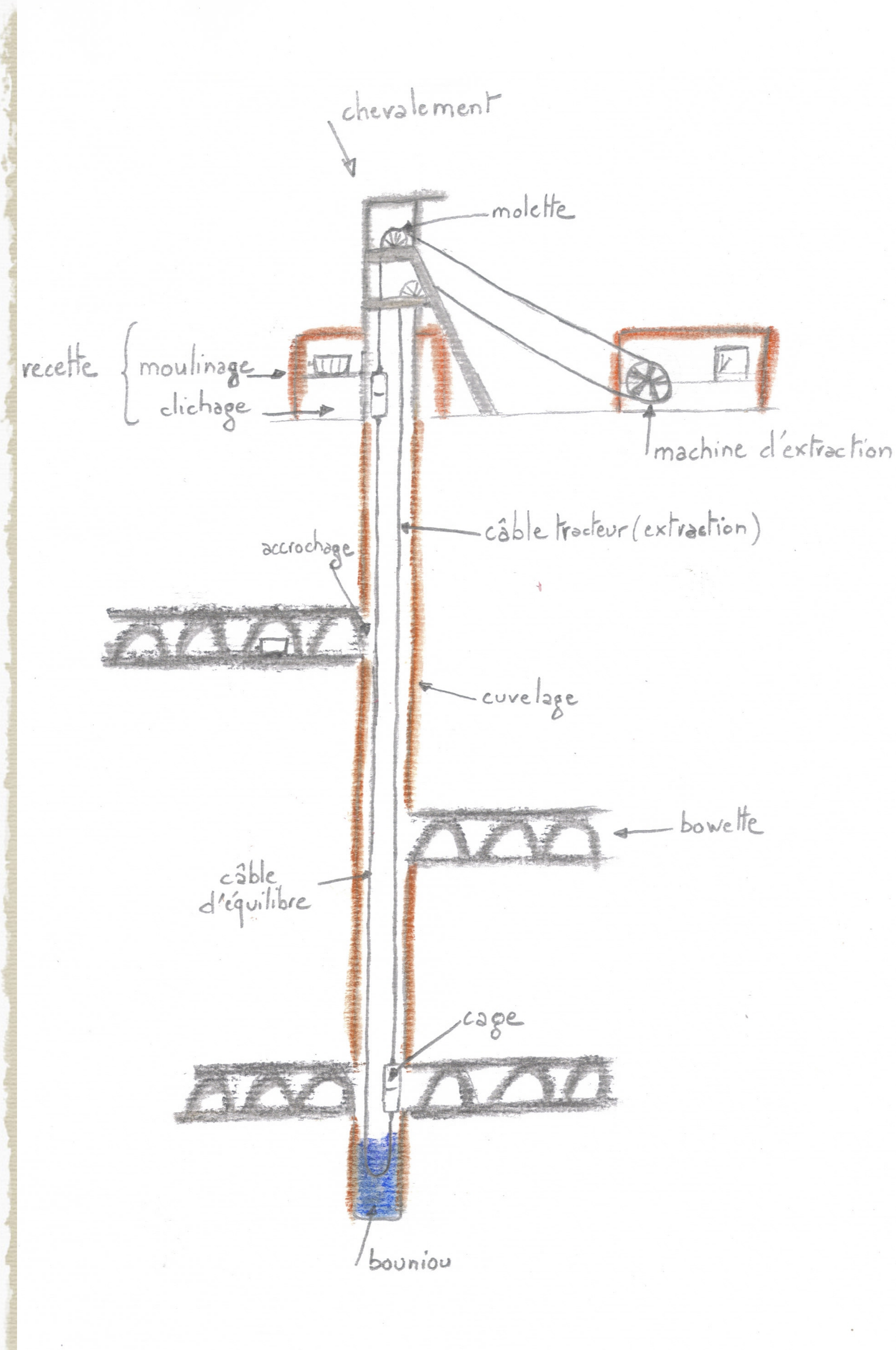
Until the middle of the 19th century, the miner descended into a cuffat, a sort of wooden half-barrel, surrounded by iron, forming both a basket for the personnel and a container for the coal, a cuffat suspended at the end of the extraction cable and circulating in the well without vertical guidance.

E. Schneider, Le Charbon (1945).

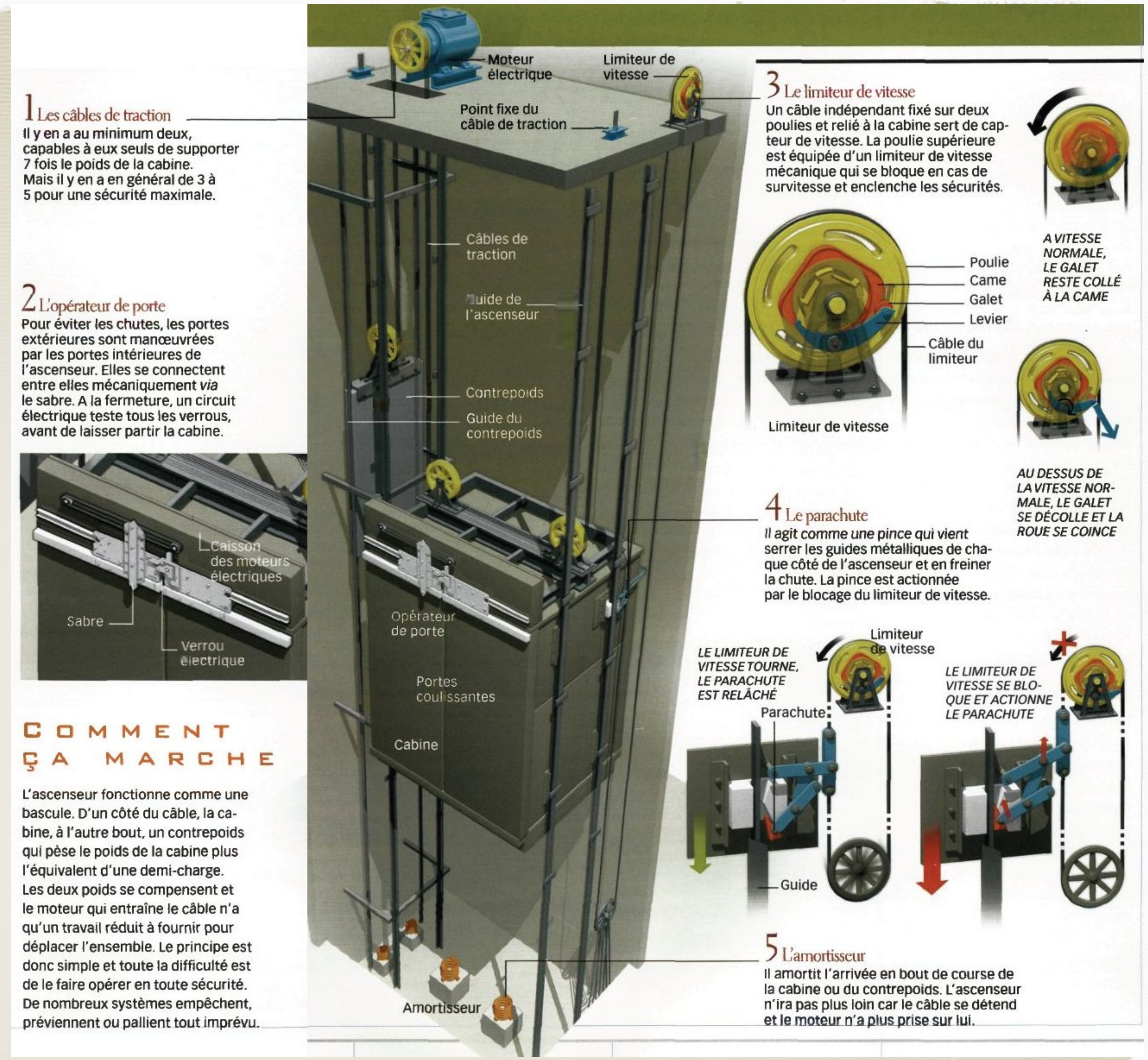
The Path of the Pendulum







Dr A. Riembault, Hygiène des ouvriers mineurs dans les exploitations houillères (Paris, 1861).



1 Les câbles de traction
Il y en a au minimum deux, capables à eux seuls de supporter 7 fois le poids de la cabine. Mais il y en a en général de 3 à 5 pour une sécurité maximale.

2 L'opérateur de porte
Pour éviter les chutes, les portes extérieures sont manœuvrées par les portes intérieures de l'ascenseur. Elles se connectent entre elles mécaniquement *via* le sabre. A la fermeture, un circuit électrique teste tous les verrous, avant de laisser partir la cabine.

3 Le limiteur de vitesse
Un câble indépendant fixé sur deux poulies et relié à la cabine sert de capteur de vitesse. La poulie supérieure est équipée d'un limiteur de vitesse mécanique qui se bloque en cas de survitesse et enclenche les sécurités.

A VITESSE NORMALE, LE GALET RESTE COLLÉ À LA CAME

AU DESSUS DE LA VITESSE NORMALE, LE GALET SE DÉCOLLE ET LA ROUE SE COINCE

4 Le parachute
Il agit comme une pince qui vient serrer les guides métalliques de chaque côté de l'ascenseur et en freiner la chute. La pince est actionnée par le blocage du limiteur de vitesse.

LE LIMITEUR DE VITESSE TOURNE, LE PARACHUTE EST RELÂCHÉ

LE LIMITEUR DE VITESSE SE BLOQUE ET ACTIONNE LE PARACHUTE

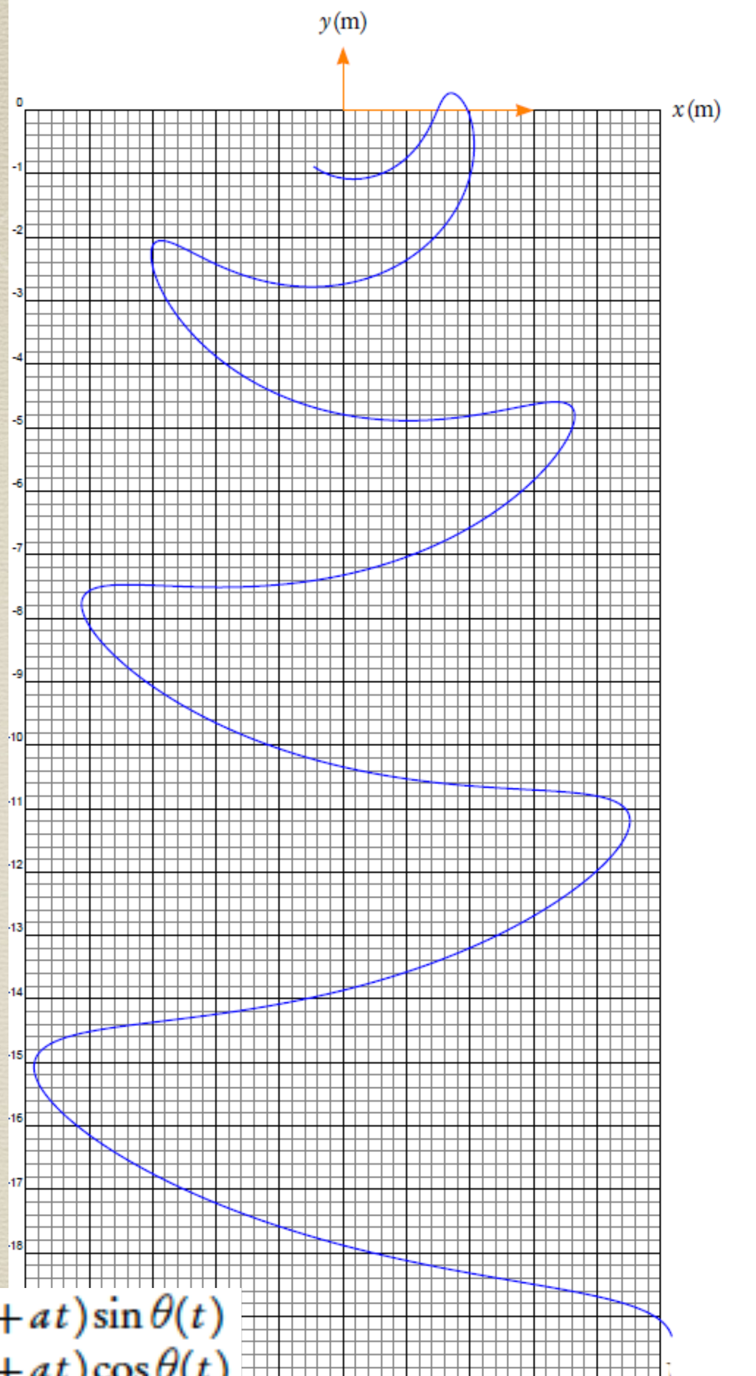
5 L'amortisseur
Il amortit l'arrivée en bout de course de la cabine ou du contrepoids. L'ascenseur n'ira pas plus loin car le câble se détend et le moteur n'a plus prise sur lui.

COMMENT ÇA MARCHE

L'ascenseur fonctionne comme une bascule. D'un côté du câble, la cabine, à l'autre bout, un contrepoids qui pèse le poids de la cabine plus l'équivalent d'une demi-charge. Les deux poids se compensent et le moteur qui entraîne le câble n'a qu'un travail réduit à fournir pour déplacer l'ensemble. Le principe est donc simple et toute la difficulté est de le faire opérer en toute sécurité. De nombreux systèmes empêchent, préviennent ou pallient tout imprévu.

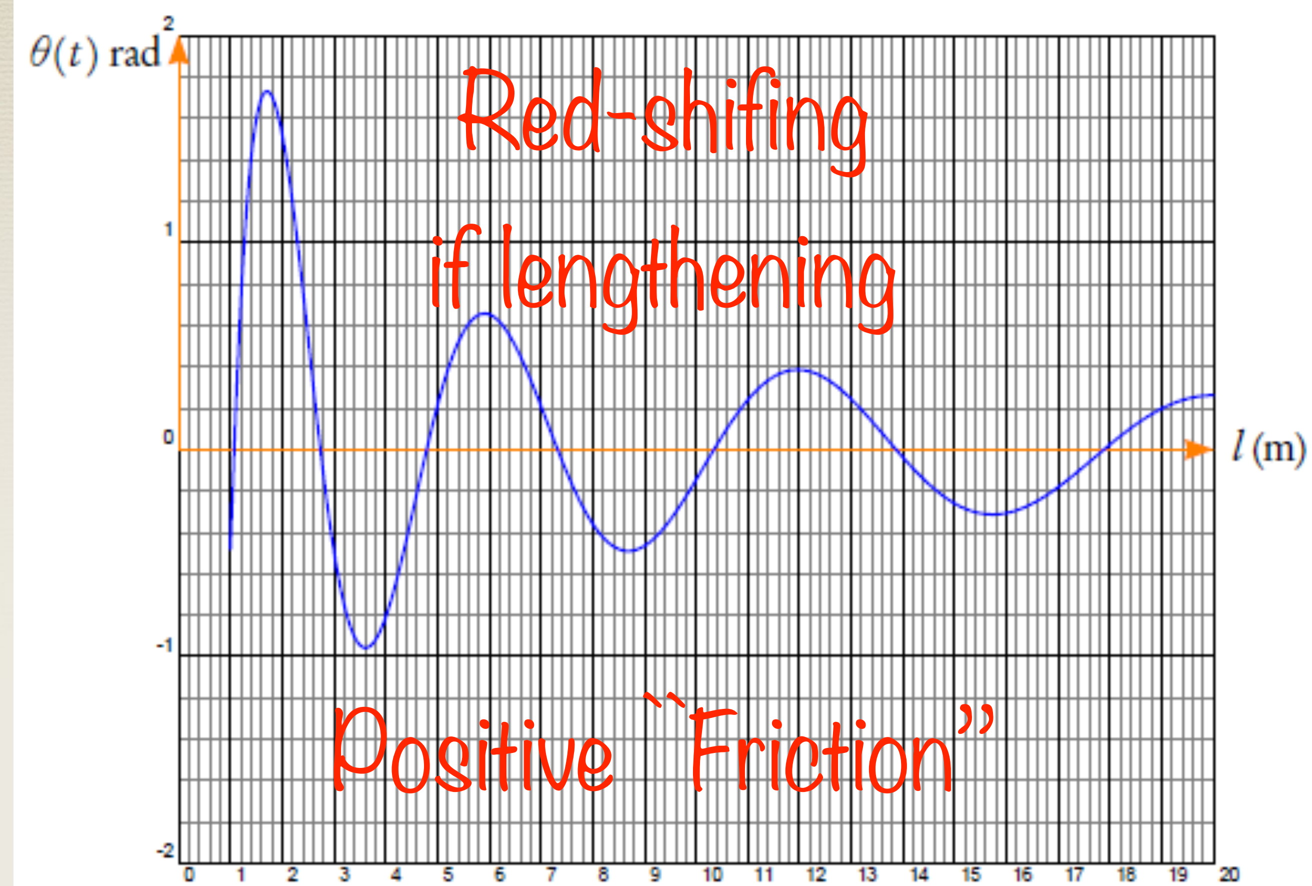
Les bennes guidées sont commodes et n'offrent pas les mêmes dangers. Dans un puits guidé, on n'a pas à redouter la rencontre des bennes ni leur accrochage contre les parois, mais il y aura les mêmes précautions à prendre pour ce qui concerne la solidité du matériel et les mesures relatives à la sureté des ouvriers.

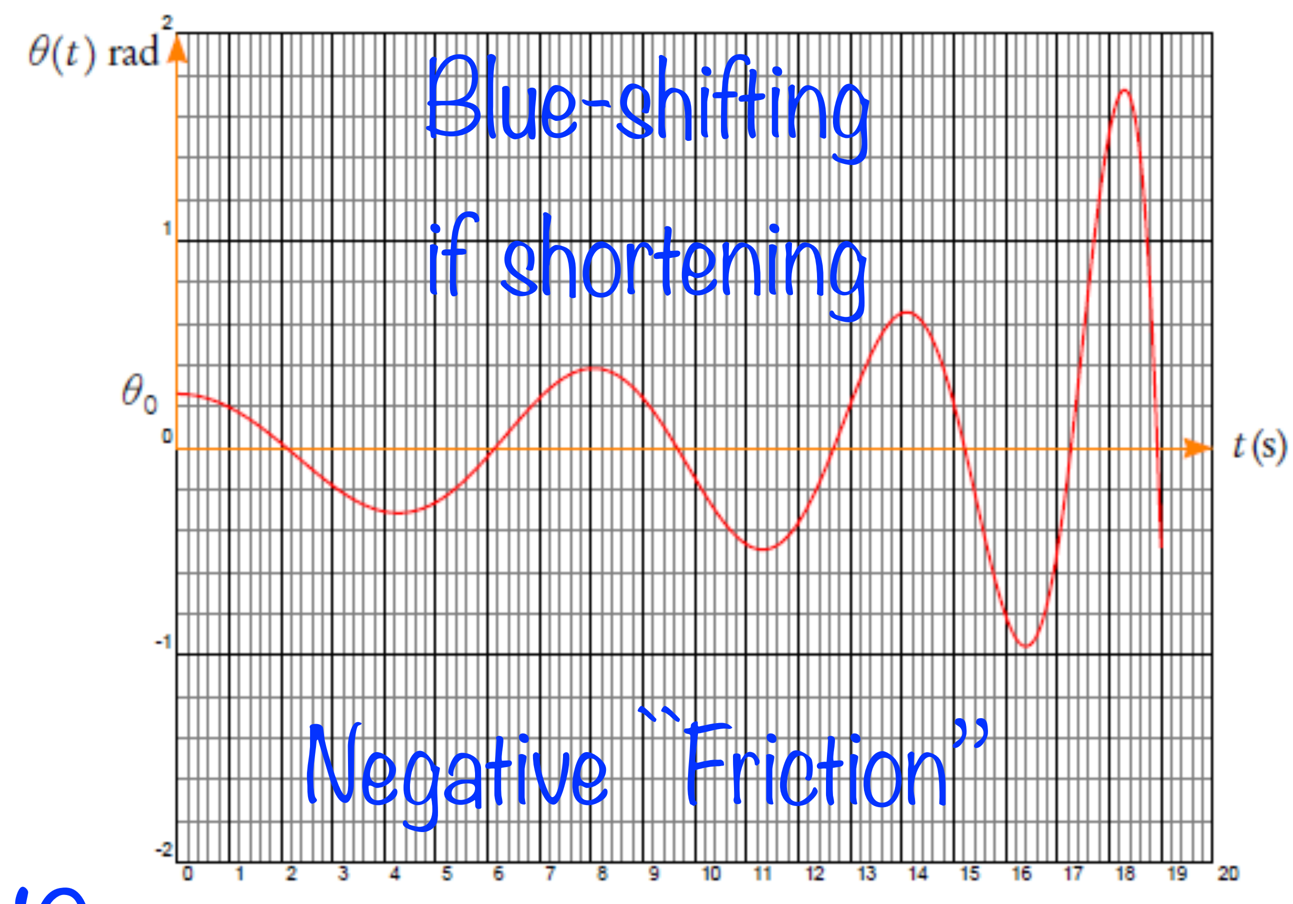
Bossut-Lemaître-Hubble ``Friction'' in Classical Mechanics



$$x = (l_0 + at)\sin\theta(t)$$
$$y = -(l_0 + at)\cos\theta(t)$$

Path of
the Pendulum
x = l(t).sinθ(t)
y = -l(t).cosθ(t)





$$\ddot{\theta} + \frac{2}{l}\dot{l}\dot{\theta} + \frac{g}{l}\sin\theta = 0 \qquad (1)$$

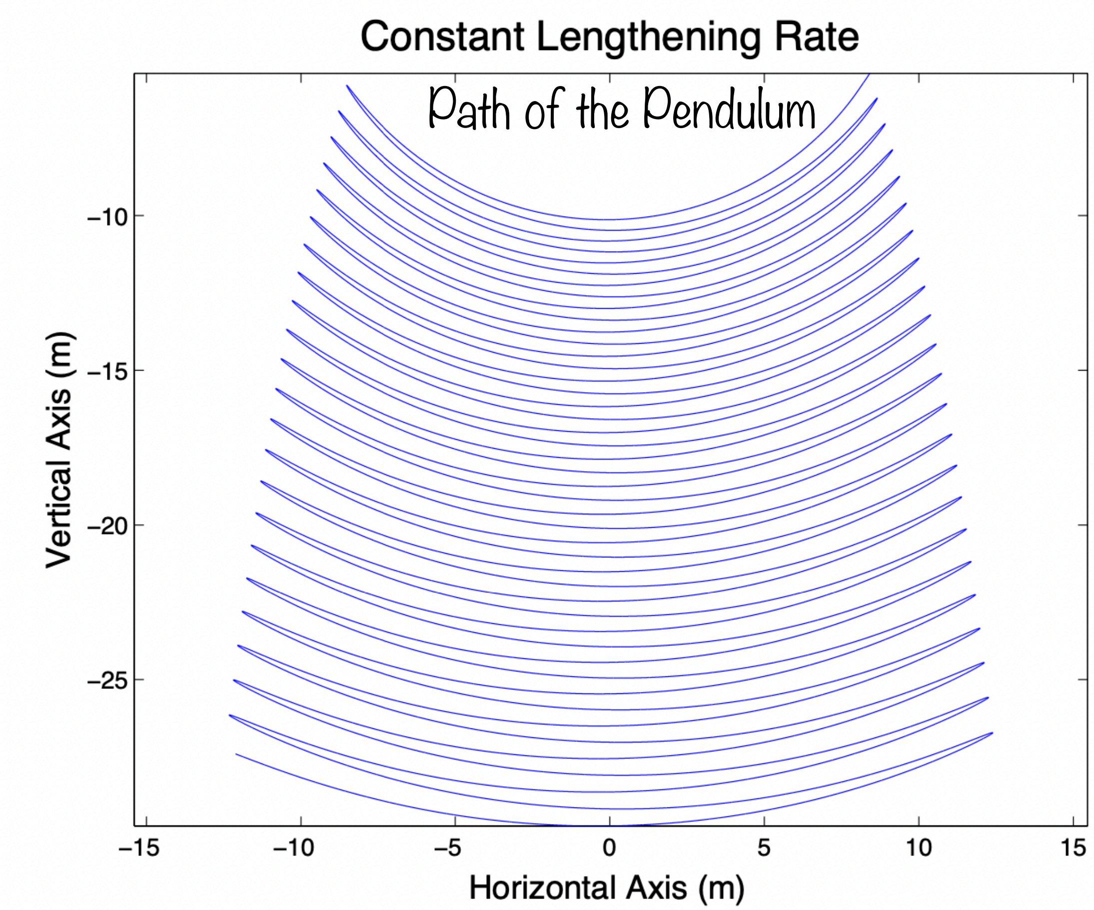
Supposons que la corde soit enroulée sur le treuil à une vitesse constante : $l = l_0 + at$. Avec $a < 0$, le charge remonte et $a > 0$ elle descend. Dans ce cas, l'équation (1) peut s'écrire :

$$\frac{d^2\theta}{dl^2} + \frac{2}{l}\frac{d\theta}{dl} + \frac{g}{a^2 l}\sin\theta = 0 \qquad (2)$$

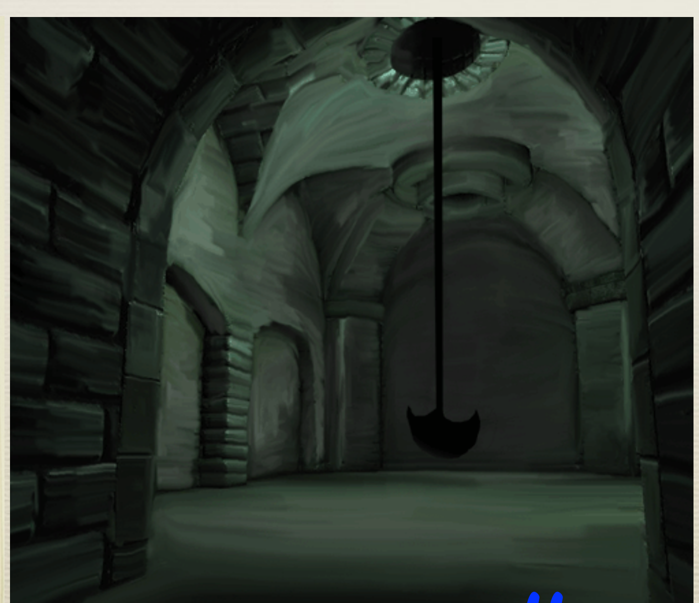
À l'instant initial $t = 0$, la corde fait un angle θ_0, petit, avec la verticale et la longueur vaut l_0. On abandonne la charge et la corde commence à s'enrouler sur le treuil à vitesse constante. Sur le graphique ci-dessous, on constate que les oscillations de la charge augmentent en amplitude au cours de la montée. Dans cet exemple, l'angle initial vaut $\theta_0 = 15°$, la longueur initiale $l_0 = 20\,\text{m}$ et la corde s'enroule autour du treuil à la vitesse de 1 m/s. La charge est remontée sur une hauteur de 19 m.

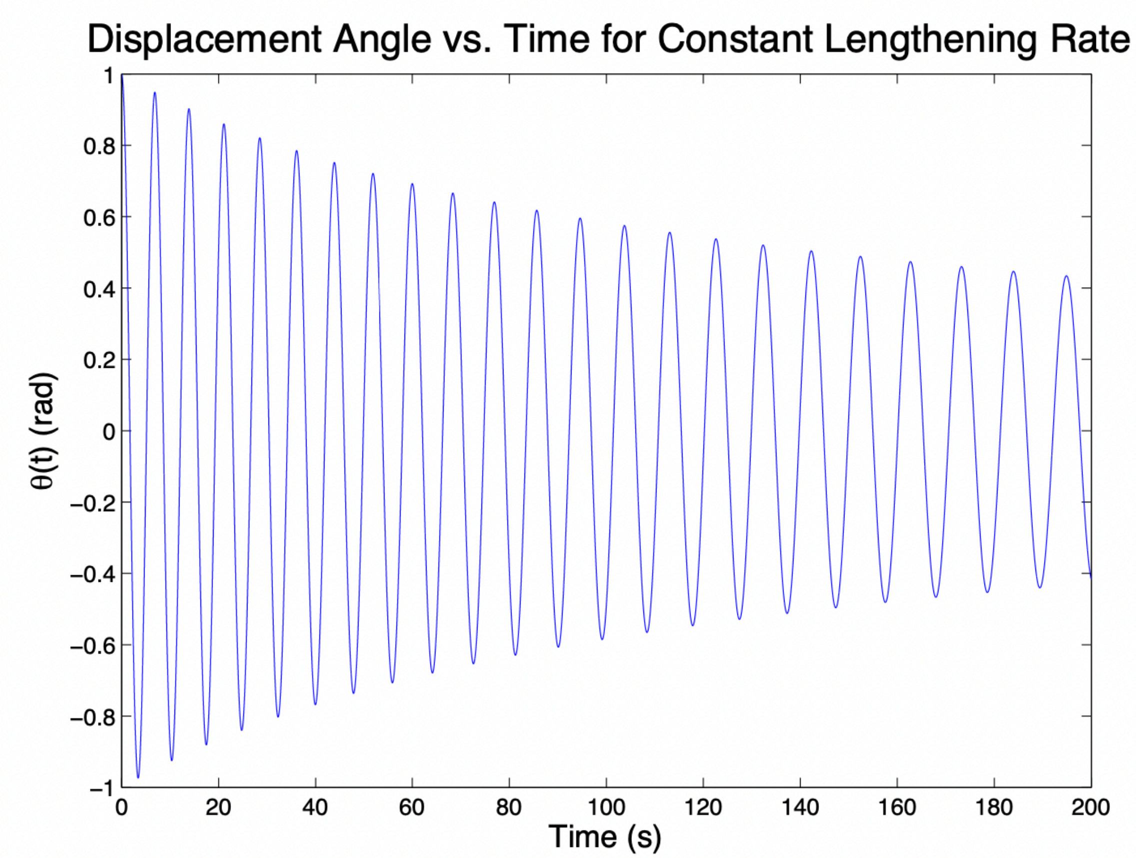
https://melusine.eu.org/syracuse/G/pstricks/pendules/

Edgar Allan Poe's Linearly Varying Length Pendulum



$$l(t) = l_0 + at$$





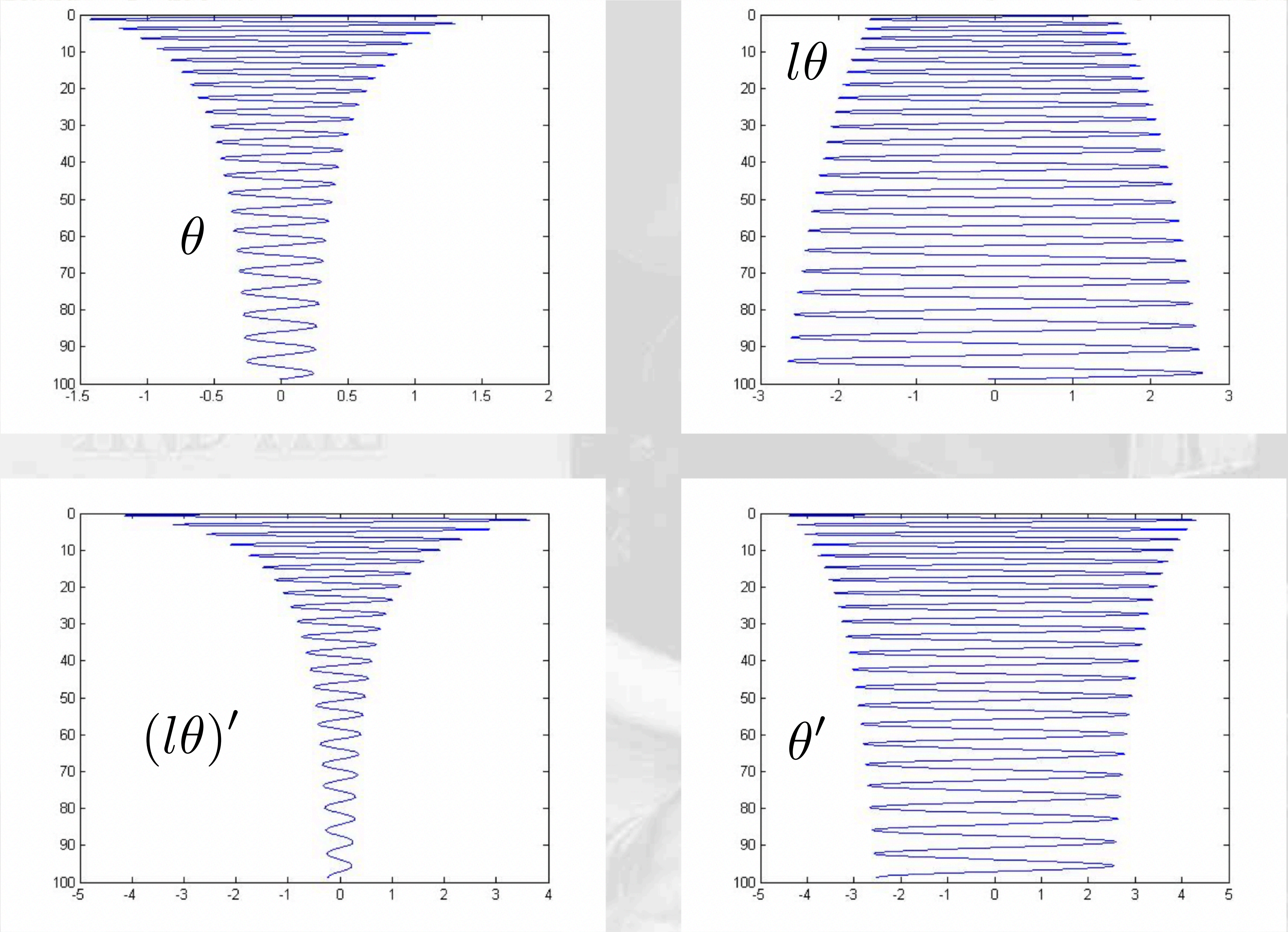


Figure 3:
From top left clockwise: 1. The Angle, 2. The Sweep,
3. The Linear Velocity, 4. The Angular Velocity.

1911 Solvay Conference

Mr. Lorentz recalls a conversation he had with Mr. Einstein some time ago, in which there was talk of a simple pendulum that could be shortened by holding the wire between two fingers and sliding towards the down. Suppose that, at the beginning, the pendulum has exactly one element of energy such that it corresponds to the frequency of its oscillations, it then seems that at the end of the experiment its energy will be less than the element which corresponds to the new frequency.

Mr. Einstein. — If we modify the length of the pendulum infinitely slowly, the energy of the oscillation remains equal to hν, if it was originally equal to hν: it varies proportionally to the frequency.

La Théorie du Rayonnement et les Quanta, report of the 1911 Solvay Conference, Gauthier-Villars, 1912, p. 450.

WKBJ Approximation

$$l\ddot{\theta} + 2\dot{l}\dot{\theta} + g\sin\theta = 0$$

$$l = l(t)$$

$$\theta << 1 \qquad l\ddot{\theta} + 2\dot{l}\dot{\theta} + g\theta \simeq 0$$

$$\theta(t) = A(t)e^{i\phi(t)} \qquad \phi(t) = \int^t \omega(t')dt'$$

$$\dot{\theta} = (\dot{A} + i\dot{\phi}A)e^{i\phi}$$

$$\dot{\theta} = (\dot{A} + i\omega A)e^{i\phi}$$

$$\ddot{\theta} = (\ddot{A} + 2i\dot{A}\dot{\phi} + i\ddot{\phi}A - \phi^2 A)e^{i\phi}$$

$$\ddot{\theta} = (\ddot{A} + 2i\dot{A}\omega + i\dot{\omega}A - \omega^2 A)e^{i\phi}$$

$$\ddot{A} + 2\left(i\omega + \frac{\dot{l}}{l}\right)\dot{A} + \left[i\left(\dot{\omega} + 2\omega\frac{\dot{l}}{l}\right) + \left(\frac{g}{l} - \omega^2\right)\right]A = 0$$

$$2\left(i\omega + \frac{\dot{l}}{l}\right)\dot{A} + \left[i\left(\dot{\omega} + 2\omega\frac{\dot{l}}{l}\right) + \left(\frac{g}{l} - \omega^2\right)\right]A \simeq 0$$

Conservation of Action (Ehrenfest's Adiabatic Theorem)

$$St = \frac{\omega l}{\dot{l}}$$

$$St >> 1$$

$$\ddot{\theta} + \omega(t)^2\theta \simeq 0$$

$$\omega^2 = g/l$$

$$\dot{A} = -\frac{\dot{\omega}}{2\omega}A$$

$$A = \frac{C}{\omega^{1/2}}$$

$$E = \frac{1}{2}mA^2\omega^2$$

$$I = E/\omega \qquad \frac{\dot{I}}{I} = 2\frac{\dot{A}}{A} + \frac{\dot{\omega}}{\omega} = 0$$

P. Ehrenfest, Over adiabatische veranderingen van een stelsel in verband met e theorie de quanta, Verslagen Kon. Akad. Amsterdam, 25 pp. 412—433 (1916).

More (Vacua) is Different (Number of Particles)

All inertial coordinates are related by a Lorentz transformation
in a flat Minkowski space-time.
Hence, all possible vacuum states are the same, and so is the number operator.
Basis modes of particle wave-function can be decomposed into ``proper vibrations''
that are classified as either of positive- or negative-frequency (or energy) types.
Different observers agree in their observations of the same number of particles.

In a general space-time, modes can no more be classified as
either of positive- or negative-frequency types.
Different sets of basis modes may be found but without a preferred one:
the notion of vacuum and number operator now depends on the chosen set.
Different observers will disagree in their observations of the number of particles.
The concepts of particle and vacuum are not global i.e. they are observer dependent.

Reflexion/Scattering of Light in a Homogeneous Space or Creation/Production of Particles in an Expanding Universe

The decomposition of an arbitrary wave function into proper vibrations is rigorous, as far as the functions of space (amplitude-functions) are concerned, which, by the way, are exactly the same as in the static universe. But it is known, that, with the latter, two frequencies, equal but of opposite sign, belong to every space function. *These two* proper vibrations cannot be rigorously separated in the expanding universe. That means to say, that if in a certain moment only one of them is present, the other one can turn up in the course of time.

Generally speaking this is a phenomenon of outstanding importance. With particles it would mean production or anihilation of matter, merely by the expansion, whereas with light there would be a production of light travelling in the opposite direction, thus a sort of reflexion of light in homogeneous space. Alarmed by these prospects, I have investigated the question in more detail. Fortunately

E. Schroedinger, The proper vibrations of the expanding universe, Physica, 6 (7-12), p. 899-912 (1939).

What we can learn from this result? In the classical case the momentum is fixed and we have two frequencies positive and negative corresponding to waves traveling in opposite directions. When initially we had only one wave propagating in a given fixed direction, then a wave propagating in opposite direction is created and the initial wave is amplified, which means that a pair of particles with opposite momenta is created. Therefore particles are created in pairs and they are moving in opposite directions. Momentum is conserved and this is natural because, due to symmetries of the background space-time, momentum has to be conserved. Since particles are created in pairs, in the case of electromagnetic field we have to create simultaneously two photons and for this we need a portion of energy equal to $2\cdot\hbar\,\omega_0$.

Y. B. Zel'dovich, Creation of particles by gravitational field, Physics of the Expanding Universe, Cracow School on Cosmology Jodłowy Dwór, September 1978 Poland, p. 60-80 (1978).

Let us now return to the classical calculations and instead of working with electromagnetic field, which has many components, we will restrict ourselves to a scalar field which satisfies the wave equation

$$\Box\varphi = \frac{\partial^2 \varphi}{\partial t^2} - \Delta\varphi = 0 \qquad (7)$$

where units are such that c = 1. The metric of the background space-time we take in the form

$$ds^2 = dt^2 - a^2(t)dx^2 - b^2(t)dy^2 - c^2(t)dz^2 \qquad (8)$$

i.e. with flat three-dimensional spatial sections t = const. Assuming that

$$\varphi = f(t)\ e^{i\vec{k}\vec{x}}, \qquad (9)$$

the wave equation reduces to

$$\frac{d^2 f}{dt^2} = -\omega^2 f, \qquad (10)$$

where

$$k_x^2\ a^{-2} + k_y^2\ b^{-2} + k_z^2\ c^{-2} = \omega^2(t). \qquad (11)$$

The general solution of (10) when $\omega = \text{const}$ is

Simple Pendulum

$$f = A\ e^{-i\omega t} + B\ e^{i\omega t} \qquad St = \frac{\omega a}{\dot{a}} \to \infty \qquad (12)$$

(For one running wave $A = A_o$, $B = 0$ one cannot define the phase by taking $e^{-i\omega t+i\theta}$ instead of $e^{-i\omega t}$ since it is equivalent to spatial displacement. The phase is important as a relation between A_o and B_o or between $A_o(k)$ and $A_o(-k)$ which brings us back to standing waves. The calculation for one running wave $A = A_o$, $B = 0$ is already equivalent to the phase-averaged calculation for standing waves.) In this simple case of homogeneous metric no mixing of modes occurs. One can now work with a single mode and we have a situation similar to mechanical oscillations with time dependent period (pendelum with slowly varying length).

As is well known if ω is slowly varying in time we can use the adiabatic approximation and replace $f = A\, e^{-i\omega t}$ by

$$f = A\, e^{-i\int \omega\, dt}. \tag{13}$$

Allowing A to be time dependent (10) leads to

$$0 = \ddot{f} + \omega^2 f = -2i\,\omega \frac{\dot{A}}{A} f - i\,\dot{\omega} f + \left(\frac{\dot{A}}{A}\right)^{\cdot} f + \left(\frac{\dot{A}}{A}\right)^2 f, \tag{14}$$

and neglecting quadratic terms and higher derivatives we get

$$\frac{\dot{A}}{A} = -\frac{1}{2}\frac{\dot{\omega}}{\omega} \tag{15}$$

This equation can be easily solved and finally we obtain

The particle number in the limit of a smooth and slow expanding universe is an adiabatic invariant. Particles are NOT created.

$$A = \frac{\alpha}{\sqrt{\omega}}$$

To the degree of approximation of E h r e n f e s t s theorem there is no mutual contamination of positive and negative frequency solutions.

Therefore the general solution of equation (10) in the adiabatic approximation is

$$St = \frac{\omega a}{\dot{a}} >> 1$$

Adiabatic Pendulum

$$f = \frac{\alpha}{\sqrt{\omega}}\, e^{-i\int \omega\, dt} + \frac{\beta}{\sqrt{\omega}}\, e^{i\int \omega\, dt} \tag{17}$$

where α and β are constants but $\omega = \omega(t)$. Now we can go further to the post adiabatic approximation and assume that α and β

depend on time. In this case the decomposition of the field into positive and negative frequencies is not unique. We can use this freedom to impose additional conditions, which are necessary to get rid of some pathological situations. This method of solving equation (10) was invented by Lagrange.

Assuming that the first time derivatives of α, β and ω are small we can compute the first time derivative of f and get

$$\dot{f} = -i\alpha\sqrt{\omega}\, e^{-i\int\omega dt} + i\beta\sqrt{\omega}\, e^{i\int\omega dt} \qquad (18)$$

This relation is true in the general case if

$$\left(\dot{\alpha} - \frac{1}{2}\frac{\dot{\omega}}{\omega}\alpha\right) e^{-i\int\omega dt} + \left(\dot{\beta} - \frac{1}{2}\frac{\dot{\omega}}{\omega}\beta\right) e^{i\int\omega dt} = 0 \qquad (19)$$

Equation (10) will be satisfied if in addition to (19) we have

$$-i\alpha\sqrt{\omega}\left(\frac{\dot{\alpha}}{\alpha} + \frac{1}{2}\frac{\dot{\omega}}{\omega}\right) e^{-i\int\omega dt} + i\beta\sqrt{\omega}\left(\frac{\dot{\beta}}{\beta} + \frac{1}{2}\frac{\dot{\omega}}{\omega}\right) e^{i\int\omega dt} = 0 \qquad (20)$$

From (19) and (20) we can compute $\dot{\alpha}$ and $\dot{\beta}$

Non-Adiabatic Pendulum

$$\dot{\alpha} = \frac{1}{2}\frac{\dot{\omega}}{\omega}\beta\, e^{2i\int\omega dt} \qquad St = \frac{\omega a}{\dot{a}} \simeq O(1) \qquad (21)$$

$$\dot{\beta} = \frac{1}{2}\frac{\dot{\omega}}{\omega}\alpha\, e^{-2i\int\omega dt}$$

This system of coupled first order differential equations has a first integral

$$|\alpha|^2 - |\beta|^2 = \text{const} \qquad (22)$$

To solve equations (21) we take as initial conditions $\alpha_{in} = 1$, $\beta_{in} = 0$, then we get in the first approximation

$$\beta_{out} = \frac{1}{2}\int \frac{\dot{\omega}}{\omega} e^{-2i\int\omega dt}\, dt \qquad (23)$$

Mean Number of Particles and Mode Mixing

Of more interest is the fact that, apart from the normalization discussed above, the initial plane wave solution, $\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega_1 t)]$, evolves into the superposition $\alpha_k \exp[i(\mathbf{k}\cdot\mathbf{x}-\omega_2 t)]+\beta_k \exp[i(\mathbf{k}\cdot\mathbf{x}+\omega_2 t)]$. A wave packet formed from a superposition of these solutions with wave numbers in a narrow range near $\mathbf{k}$ moves at early times in the $+\mathbf{k}$ direction. As a result of the expansion (or contraction) of the universe, at late times the wave packet has split into two parts, one moving in the $+\mathbf{k}$ direction and the other in the $-\mathbf{k}$ direction. Ignoring the changes in normalization caused by the effect of the expansion on the wavelength and frequency of the wave, one can say that the wave moving in the $+\mathbf{k}$ direction has grown in intensity from the value 1 to the value $|\alpha_k|^2 = 1+|\beta_k|^2$. At the same time a backward moving wave has been created with an intensity $|\beta_k|^2$.

This can be interpreted in the following heuristic way (rigorously justified in Section 3). If before the expansion of the universe the mean number of particles moving in the $+\mathbf{k}$ direction is 1, then after the expansion the mean number moving in the $+\mathbf{k}$ direction is $1+|\beta_k|^2$, and the mean number moving in the $-\mathbf{k}$ direction is $|\beta_k|^2$. (For simplicity, suppose that the particle and antiparticle of the field are the same, as for neutral mesons.)

L. Parker, Particle creation in expanding universes, Physical Review Letters, 21 (8), p. 562 (1968).

L. Parker, Particle Creation by the Expansion of the Universe, Fundamentals of cosmic physics, 7, p. 201-239 (1982).

Mutual Adulteration

culpable alteration with regard to a given system of values and norms

We have quite intentionally called *one* proper vibration the terms containing one particular spatial function ω, but *both* solutions of $f(t)$. The latter correspond to what with $R = \text{Const.}$ would be $\cos 2\pi\nu t$ and $\sin 2\pi\nu t$; or, alternatively to $e^{2\pi i\nu t}$ and $e^{-2\pi i\nu t}$. Of course the two parts keep clear of each other also in the general case. But for assigning a quite general physical meaning to this separation, one would have to know, that an $f(t)$ which during a period of constant R (or very slowly varying R) had the form (or approximately the form) $e^{2\pi i\nu t}$ will re-assume (or approximately re-assume) the form $Ae^{2\pi i\nu t}$ — and *not* $Ae^{2\pi i\nu' t} + Be^{-2\pi i\nu' t}$ — whenever $R(t)$, after an intermediate period of arbitrary variation, returns to constancy (or to approximate constancy). I can see no reason whatsoever for $f(t)$ to behave rigorously in this way, and indeed I do not think it does. There will thus be a mutual adulteration of positive and negative frequency terms in the course of time, giving rise to what in the introduction I called "the alarming phenomena".

The Schroedingers relax. Mrs. March (centre) is wife of the professor's Austrian colleague.

Schrodinger was offered a teaching position at Princeton but after extensive negotiations he declined the position. It is thought that Princeton would not house him with his wife and his mistress. Instead, he went to DUBLIN, to the Dublin Institute for Advanced Studies, who obligingly provided said housing arrangement.

While in Ireland, Schrodinger also fathered two children, by two different women.

E. Schroedinger, The proper vibrations of the expanding universe, Physica, 6 (7-12), p. 899-912 (1939).

Matching the Different Vacua

A scalar field in an expanding flat space

$$0 = \ddot{\phi} - \frac{\Delta}{a^2}\phi$$

$$\omega_k = \frac{\sqrt{-h_k}}{a} = \frac{|k|}{a}$$

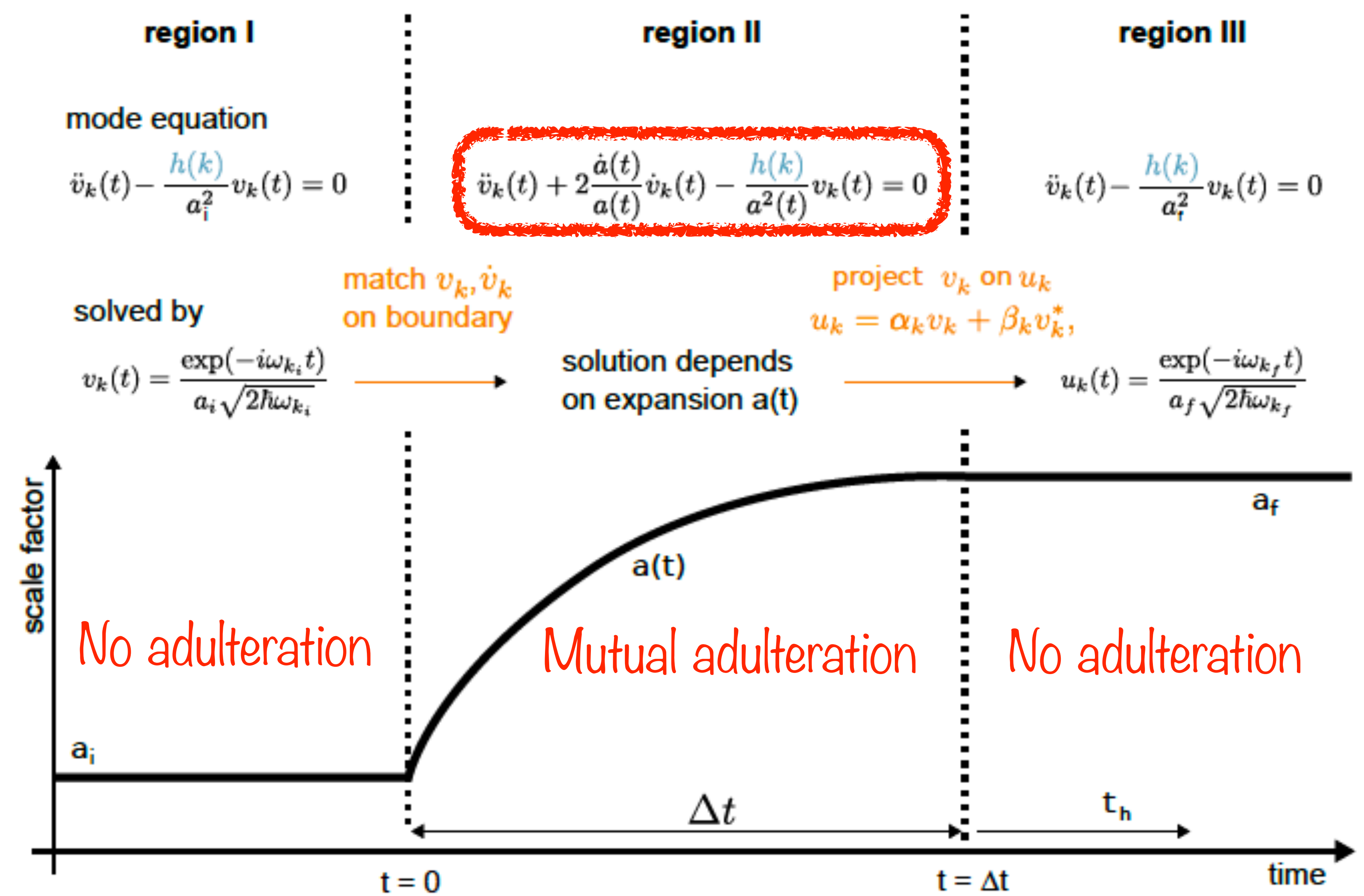


Figure 9.1: Expansion scenario for particle production and a schematic sketch of the calculation. The scenario for particle production starts and ends with a static space (region I and III). Between these regions, space expands/contracts during a period Δt. The important quantity for particle production is the evolution of the mode functions. In regions I and III, modes oscillate. In region II, the evolution is described by the mode equation and depends on the expansion itself. On the boundaries between regions, the mode functions and their derivatives must be matched. This connects mode functions in region I and region III, which in turn defines a Bogoliubov transformation between creation and annihilation operators in these regions.

Celia Viermann, Cosmological particle production and curved spaces in an ultracold quantum gas, Doctoral dissertation, June (2022).

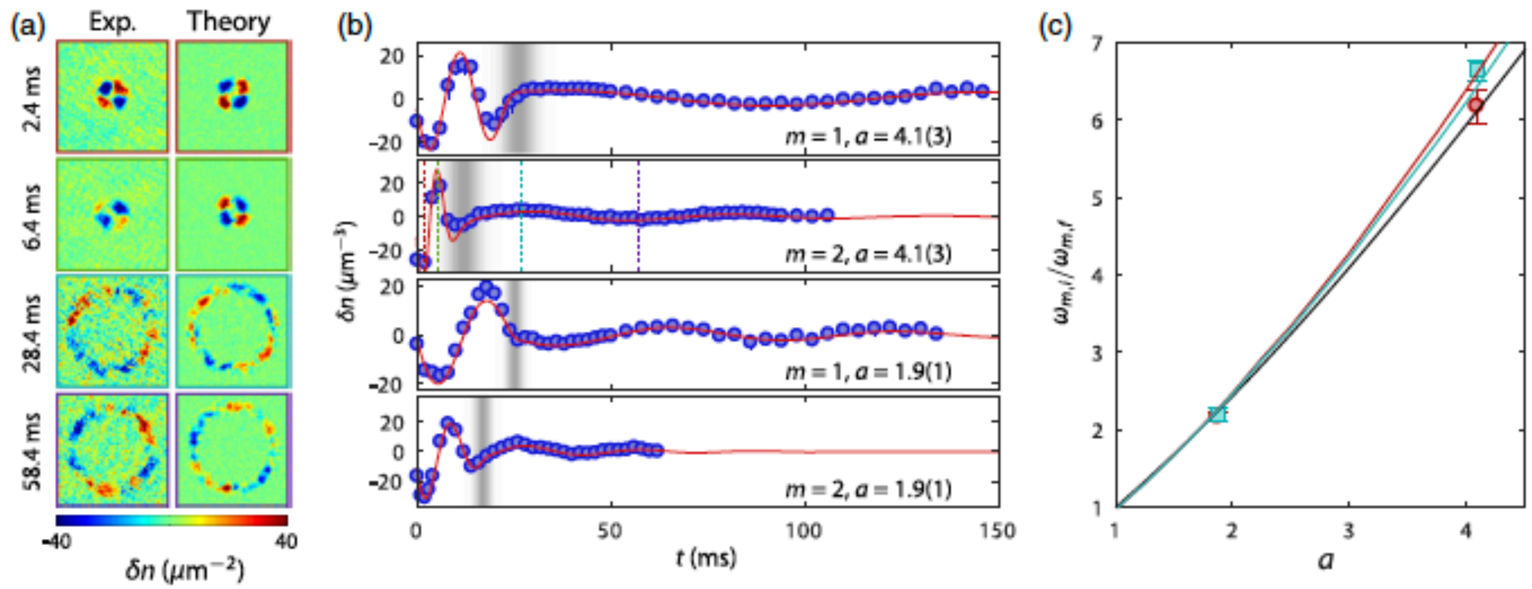


FIG. 2. Redshift of long-wave excitations. (a) Atomic density difference δn at various times for both experiment and simulation for a mode number $m = 2$ and scale factor $a = 4.1$. The density scale of images after expansion is multiplied by 10. (b) Phonon amplitude vs time for various a and m. The grey bands indicate the time during which the BEC is inflated; their intensity denotes the expansion velocity relative to that expansion's maximum. Vertical dashed lines in the $m = 2$, $a = 4.1$ panel indicate the times shown in (a). (c) Ratio of initial to final frequency vs scale factor a. Red circles indicate $m = 1$ modes; cyan squares, $m = 2$. The solid black curve is the $a^{9/7}$ expectation, and the colored curves (with mode numbers matching points) are the result of full Bogoliubov calculation.

PHYSICAL REVIEW LETTERS **128,** 090401 (2022)

Editors' Suggestion

Accurate Determination of Hubble Attenuation and Amplification in Expanding and Contracting Cold-Atom Universes

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In the expanding universe, relativistic scalar fields are thought to be attenuated by "Hubble friction," which results from the dilation of the underlying spacetime metric. By contrast, in a contracting universe this pseudofriction would lead to amplification. Here, we experimentally measure, with fivefold better accuracy, both Hubble attenuation and amplification in expanding and contracting toroidally shaped Bose-Einstein condensates, in which phonons are analogous to cosmological scalar fields. We find that the observed attenuation or amplification depends on the temporal phase of the phonon field, which is only possible for nonadiabatic dynamics. The measured strength of the Hubble friction disagrees with recent theory [Gomez Llorente *et al.*, Phys. Rev. A **100**, 043613 (2019) and Eckel *et al.*, SciPost Phys. **10**, 64 (2021)].

DOI: 10.1103/PhysRevLett.128.090401

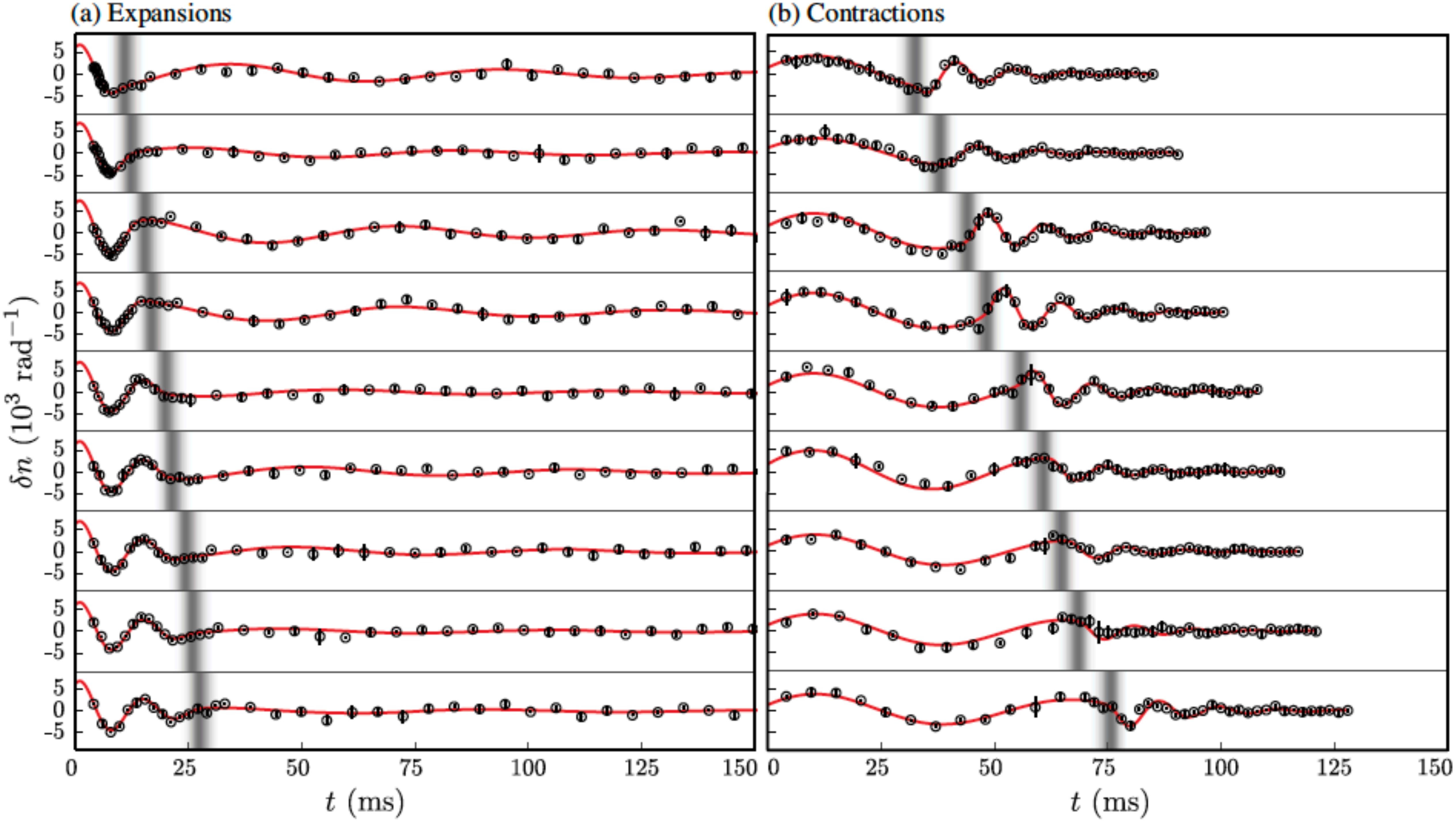


FIG. 3. Phonon amplitude δn as a function of time t for (a) expanding and (b) contracting tori. The symbols, curves, and grayscale bars are all as notated in Fig. 2(d). The expansion data (a) used $R_i = 11.9(2)$ μm and $R_f = 38.4(6)$ μm, and vice versa for contraction (b). The red curves show simultaneous fits to a complete dataset, which includes all expansions or contractions.

Conservation de l'Action d'Onde

$$I = E/\omega$$

$$\frac{dI}{dt} = 0$$

$$\frac{D\mathcal{A}}{Dt} = 0$$

$$\frac{\partial \mathcal{A}}{\partial t} + \nabla.\left(\mathcal{A}\mathbf{c}_g\right) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{E}{\omega - \mathbf{k}.\mathbf{U}}\right) + \nabla.\left[\left(\frac{E}{\omega - \mathbf{k}.\mathbf{U}}\right)\mathbf{c}_g\right] = 0$$

$$\frac{Ec_g}{\omega - \mathbf{k}.\mathbf{U}} = cste$$

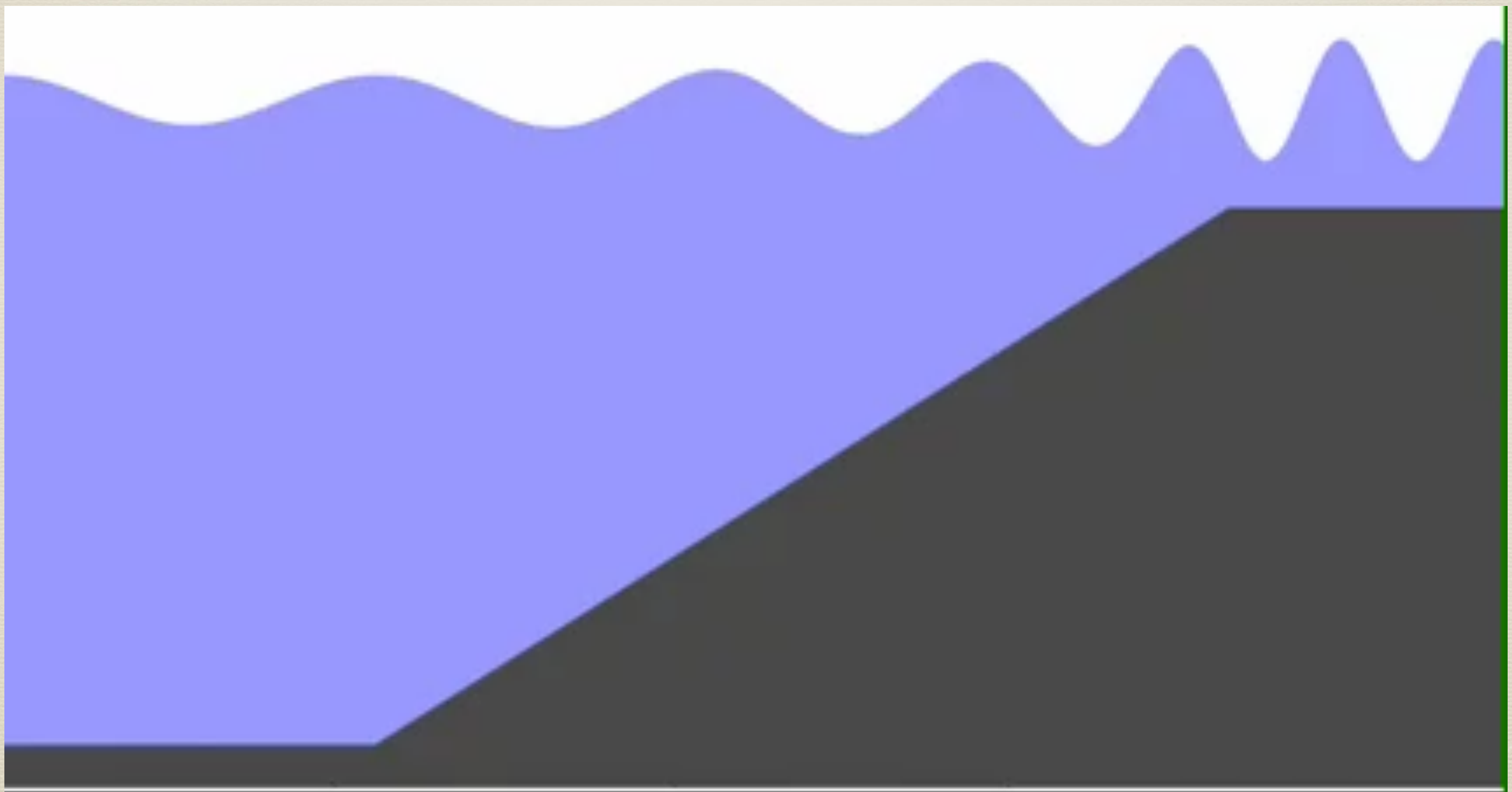
F. P. Bretherton and C. J. Garrett, Wavetrains in inhomogeneous moving media, Proc. Roy. Soc. A. 302, p. 529-554 (1968)

Green's Law (1838)

No flow current.

Changing bathymetry.

$$Ec_g = cste$$



$$E = \frac{1}{2}\rho g a^2 \qquad \lambda = T\sqrt{gh} \qquad A_x = A_0 \left(\frac{W_0}{W_x}\right)^{1/2} \left(\frac{h_0}{h_x}\right)^{1/4}$$

G. Green, On the motion of waves in a variable canal of small width and depth,
Transactions of the Cambridge Philosophical Society, Vol. VI, Part IV., p. 457 (1838).

Waves in shallow water are described by the linearised shallow water equations

$$\frac{\partial u}{\partial t} + g\frac{\partial \eta}{\partial x} = 0, \quad \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(h_0 u) = 0, \tag{10}$$

where $u(x,t)$ is the horizontal fluid velocity, and $h(x,t) = h_0(x) + \eta(x,t)$ is the water depth. The mean depth with a flat free surface is $h_0(x)$, varying to reflect the bottom topography, and $\eta(x,t)$ is the (small) displacement of the free surface.

Eliminating u between the two equations (10), we obtain a modified wave equation for $\eta(x,t)$,

$$\frac{\partial^2 \eta}{\partial t^2} = g\frac{\partial}{\partial x}\left(h_0(x)\frac{\partial \eta}{\partial x}\right), \tag{11}$$

which resembles the generalised Klein–Gordon equation from previous lectures, with $\alpha^2 = gh_0(x)$ and $\beta = 0$. Equation (11) may be rewritten as

$$\frac{\partial^2 \eta}{\partial t^2} = gh_0(x)\frac{\partial^2 \eta}{\partial x^2} + g\frac{\partial h_0}{\partial x}\frac{\partial \eta}{\partial x}, \tag{12}$$

where the last term is the "form drag" due to a sloping bottom. If we assume time-harmonic waves with $\eta(x,t) = y(x)\cos(\omega t)$, and substitute $h_0(x) = \beta x$, we obtain an ODE for the spatial variation,

$$x\frac{d^2 y}{dx^2} + \frac{dy}{dx} + \frac{\omega^2}{g\beta}y = 0. \tag{13}$$

Putting $x = \frac{1}{4}(g\beta/\omega)\, s^2$ transforms this ODE into Bessel's equation of order 0,

$$\frac{d^2 y}{ds^2} + \frac{1}{s}\frac{dy}{ds} + y = 0, \tag{14}$$

with solution as $y = aJ_0(s)$, where a is an arbitrary constant. In terms of the original variable x,

$$y(x) = aJ_0\left(\frac{2\omega}{\sqrt{g\beta}}x^{1/2}\right). \tag{15}$$

The other solution of Bessel's equation has a logarithmic singularity as $x \to 0$.

From lecture 10, we recall that the asymptotic behaviour of J_0 for large argument is given by

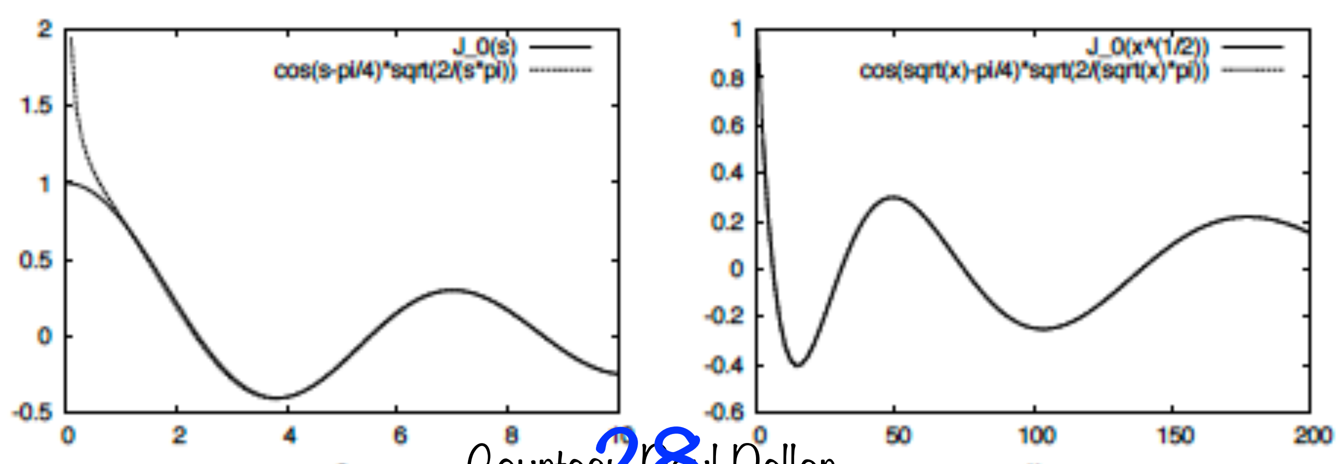
$$J_0(\xi) \sim \sqrt{\frac{2}{\pi\,\xi}}\cos(\xi - \pi/4) \quad \text{as } \xi \to \infty. \tag{16}$$

The complete solution is thus given by

$$\eta(x,t) \sim a\left(\frac{\sqrt{g\beta}}{\omega\pi}\right)^{1/2} x^{-1/4}\cos\left(\frac{2\omega}{\sqrt{g\beta}}x^{1/2} - \pi/4\right)\cos(\omega t) \text{ as } x \to \infty, \tag{17}$$

with the promised $x^{-1/4}$ behaviour of the amplitude of the free surface displacement.

In fact, the stationary phase approximation in (16) is remarkably accurate for $\xi > 1$ (see left picture).



Courtesy Paul Dellar

The equation of motion is derived from the Lagrangian equation

$$\frac{d}{dL}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}, \tag{117}$$

where $L = K - V$ is the Lagrangian, given by

$$L = \frac{1}{2}m\ell^2\dot{\theta}^2 - \frac{1}{2}mg\ell\,\theta^2 + \frac{1}{2}mh^2 - mg(\ell_0 - \ell). \tag{118}$$

From this expression we reach the equation of motion

$$\ell\ddot{\theta} + 2h\dot{\theta} = -g\theta, \tag{119}$$

valid for small oscillations. Changing variable from t to $\ell = \ell_0 + ht$ this equation becomes

$$\ell\frac{d^2\theta}{d\ell^2} + 2\frac{d\theta}{d\ell} + \frac{g}{h^2}\theta = 0, \tag{120}$$

which is the equation derived by Lecornu [64].

It is convenient to define the variable $s = \ell/g$ or $s = at + b$, where $a = h/g$ and $b = \ell_0/g$, from which we may write the equation of motion as

$$s\frac{d^2\theta}{ds^2} + 2\frac{d\theta}{ds} + \frac{\theta}{a^2} = 0. \tag{121}$$

It is convenient to define the variable $s = \ell/g$ or $s = at + b$, where $a = h/g$ and $b = \ell_0/g$, from which we may write the equation of motion as

$$s\frac{d^2\theta}{ds^2} + 2\frac{d\theta}{ds} + \frac{\theta}{a^2} = 0. \tag{121}$$

Performing the change of variables defined by $z = 2\sqrt{s}/a$ and $\phi = z\theta$ we reach the equation

$$z^2\frac{d^2\phi}{dz^2} + z\frac{d\phi}{dz} + (z^2 - 1)\phi = 0. \tag{122}$$

In this form, we see that the solutions are the Bessel functions of first order $J_1(z)$ and $Y_1(z)$ [68], that is,

$$\phi = A_1 J_1(z) + A_2 Y_1(z), \tag{123}$$

where A_1 and A_2 are constant.

$$\theta = \frac{1}{z}[A_1 J_1(z) + A_2 Y_1(z)], \tag{124}$$

which gives θ as a function of t if we recall that $z = 2\sqrt{s}/a$ and that $s = at + b$.

As we wish to get the solution for a very slow variation of the length of the pendulum, which means that a is very small, it suffices to consider the solution for large values of z. For as $z = 2\sqrt{s}/a$ and considering a finite value of $s = at + b$, z will increase as $1/a$. Therefore, we use the asymptotic expression of the Bessel functions [68], as did Trutkov and Fock [66], namely

$$J_1(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sin\left(z - \frac{\pi}{4}\right), \tag{125}$$

$$Y_1(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \frac{\pi}{4}\right). \tag{126}$$

The solution can thus be written as

$$\theta = cs^{-3/4}\cos\left(\frac{2}{a}\sqrt{s} - \frac{\pi}{4}\right). \tag{127}$$

The energy E of the pendulum is the first part of the kinetic energy given by (115) plus the first part of the potential energy given by (116),

$$E = \frac{m}{2}\left(\ell^2\dot{\theta}^2 + \ell g\theta^2\right), \tag{128}$$

which can be written as

$$E = \frac{m}{2}g^2\left[a^2s^2\left(\frac{d\theta}{ds}\right)^2 + s\theta^2\right]. \tag{129}$$

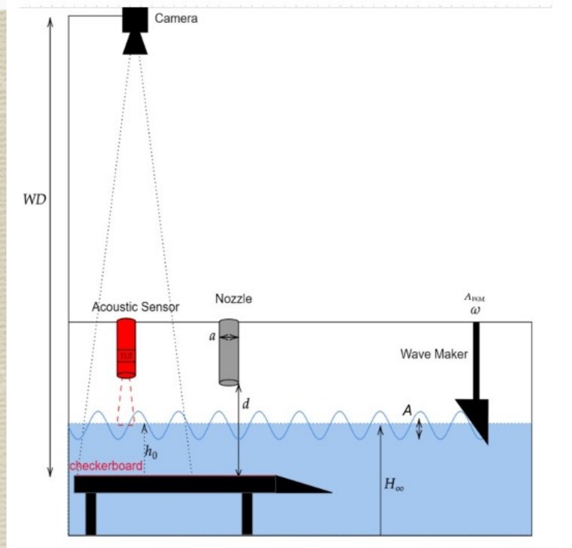
Replacing the solution (127) in this equation, we reach the following expression for the energy

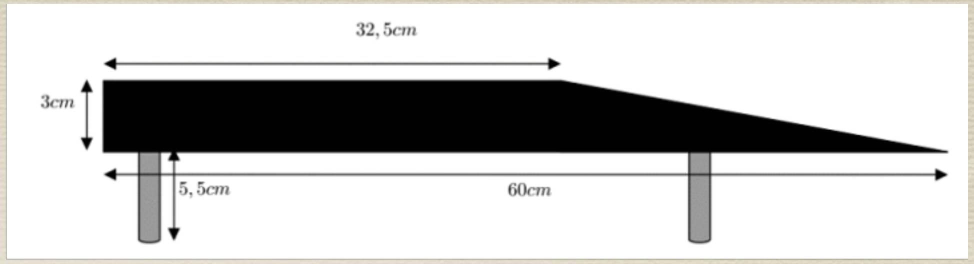
$$E = \frac{mg^2c^2}{2\sqrt{s}} = \frac{mgc^2}{2}\sqrt{\frac{g}{\ell}}, \tag{130}$$

where we have neglected terms of order equal or larger that a. That is, the energy of the pendulum is proportional to the inverse of $\sqrt{\ell}$, the Rayleigh relation. Bearing in mind that the frequency is $\omega = \sqrt{g/\ell}$, we may wright

$$E = \frac{mgc^2}{2}\omega, \tag{131}$$

and E/ω is an adiabatic invariant.





$$h_\infty = 4,3\ mm \quad T = 0,2\ s \quad A = 10\ mm$$

From Thibault Mergault Master Thesis under the supervision of Germain Rousseaux (Poitiers University, 2023).

Inflation with a Scalar Field for a Constant Light Speed

$$\frac{\dot{a}}{a} \simeq cst$$

$$S_\phi = \int d^4x\sqrt{-g}\left[-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\right]$$

$$\begin{aligned}\delta S_\phi &= \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu(\delta\phi)\times 2 - V'(\phi)\delta\phi\right]\\ &= \int d^4x\left[\partial_\nu\left(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\right)(\delta\phi) - \sqrt{-g}V'(\phi)\delta\phi\right]\\ &= \int d^4x\sqrt{-g}\left[\frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\right) - V'(\phi)\right]\delta\phi\end{aligned}$$

$$\frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi\right) - V'(\phi) = 0$$

$$g_{\mu\nu} = diag(-1,\ a^2(t),\ a^2(t),\ a^2(t))$$

$$\sqrt{-g} = a^3(t)$$

$$\tfrac{1}{a^3}\partial_t\left(-a^3\dot{\phi}\right) + \tfrac{1}{a^3}\times a^3\times a^{-2}\nabla^2\phi - V'(\phi) = 0$$

$$\ddot{\phi} + 3\tfrac{\dot{a}}{a}\dot{\phi} - \tfrac{1}{a^2}\nabla^2\phi + V'(\phi) = 0$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

Physics of bubble oscillations

Werner Lauterborn and Thomas Kurz

For an analytical treatment of nonspherical bubble oscillation one usually assumes a nearly spherical bubble with the surface perturbed only slightly to allow for a linearization of the fluid-dynamical equations around the state of spherical symmetry. The radial dynamics then is described to a good approximation by the well-known, previously discussed models, for example, the Rayleigh–Plesset equation (2). Let $r_S(\Theta, \varphi; t)$ denote the location of the surface in a spherical coordinate system with its origin at the bubble centre, and $R(t)$ be the time-dependent radius of the associated sphere (figure 44). The surface perturbation is expanded into spherical harmonics,

$$r_S(\Theta, \varphi; t) = R(t) + \sum_{n=1}^{\infty} \sum_{m=-n}^{n} a_{n,m}(t) Y_{n,m}(\Theta, \varphi), \qquad (24)$$

with time-dependent complex coefficients $a_{n,m}(t)$. The spherical harmonics are given by

$$Y_{n,m} = C_{n,m} P_n^{(m)}(\cos\Theta) \exp(im\varphi), \qquad (25)$$

where $C_{n,m}$ denotes a normalization factor and $P_n^{(m)}$ is the associated Legendre polynomial of the first kind. Each spherical harmonic describes a surface mode with an oscillation pattern having $2m$ node lines in the azimuthal (φ-)direction and n nodes in the polar (Θ-)direction. The coefficients $a_{n,m}$ are the oscillation amplitudes of the corresponding modes. For small amplitudes $a_{n,m}$ the linearized fluid-dynamical equations yield a system of mutually uncoupled, linear ordinary differential equations that are parametrically coupled to the radial dynamics $R(t)$:

$$\ddot{a}_{n,m} + 3\frac{\dot{R}}{R}\dot{a}_{n,m} + (n-1)\left(\frac{\sigma(n+1)(n+2)}{\varrho R^3} - \frac{\ddot{R}}{R}\right) a_{n,m} = 0. \qquad (26)$$

This model assumes incompressible, irrotational and inviscid flow and thus does not take into account damping due to viscosity. Inclusion of dissipation and vorticity gives a by far more complicated description involving integro-differential equations not considered here.

Oscillations or Instability

A proof of this analogy in the linear regime is easy to obtain. In the classical paper in Ref. [4], Plesset develops the linear perturbation equations of motion for an almost spherical surface between two incompressible fluids,

$$\ddot{a}_n + 3\frac{\dot{R}}{R}\dot{a}_n + W_n a_n = 0. \quad (1)$$

Here R is the mean radius, and a_n denotes the amplitude of some mode with angular momentum number n. Denote by σ the surface tension coefficient, and by $\rho_{in/out}$ the densities inside and outside the bubble, respectively. W_n takes the form

$$W_n = \frac{O(\frac{n}{R})(-\ddot{R})(\rho_{out} - \rho_{in}) + O((\frac{n}{R})^3)\sigma}{\rho_{in} + \rho_{out}}. \quad (2)$$

For large n, a_n dynamics are much more rapid than those of the radius, which can be regarded as quasistatic. Thus, for $W_n > 0$, the equation describes a damped/pumped harmonic oscillator. Choice of damping or pumping is determined by the sign of $\dot{R}$. For empty bubbles, plugging in $\rho_{in} = 0$ we find that the oscillation frequency, $\sqrt{W_n}$, follows the well-known gravity-capillary spectrum $\omega_k = \sqrt{gk + \alpha k^3}$. Here k is the wave number, g is gravity's constant acceleration, and $\alpha \equiv \sigma/\rho$, is the capillary coefficient. For bubbles we simply plug in $g = -\ddot{R}$, which is the fictitious acceleration exerted on an observer moving with the radius, and $k \sim n/R$.

If acceleration and density gradient are of different signs, and surface tension is small enough, it might be that $W_n < 0$. In such a case, perturbation growth is exponential with the rate $\sqrt{gk}$, corresponding to the linear growth rate of Rayleigh-Taylor instabilities.

P. Emanuel and A. Feigel, Turbulence and capillary waves on bubbles, Physical Review E, 104 (2), 025108 (2021).

Macroscopic acousto-mechanical analogy of a microbubble

Jennifer Chaline,[1] Noé Jiménez,[2] Ahmed Mehrem,[2] Ayache Bouakaz,[3] Serge Dos Santos,[4,5] and Víctor J. Sánchez-Morcillo[2]

In the presence of an acoustic field, microbubbles can be forced to oscillate in different ways. The most common oscillation mode is the radial mode (Fig. 1), where the bubble compresses and expands radially, maintaining its spherical shape. The basic model describing the radial dynamics of a bubble is the Rayleigh-Plesset equation for the time dependent radius $R(t)$

$$\rho\left(R\ddot{R}+\frac{3}{2}\dot{R}^2\right)=P_g-P_0-P_A(t)-\frac{2\sigma}{R}-4\mu\frac{\dot{R}}{R}, \quad (1)$$

with ρ the density of the surrounding fluid, $P_g = \left(P_0+\frac{2\sigma}{R_0}\right)\left(\frac{R_0}{R}\right)^{3\gamma}$, the gas pressure inside the bubble, P_0 is the hydrostatic pressure, R_0 is the equilibrium radius of the bubble, P_A the acoustic pressure, σ the surface tension of the bubble, μ the dynamic viscosity and γ the polytropic exponent. Some generalizations of this model

$$\ddot{a}_n+\left(\frac{3\dot{R}}{R}+\frac{2(n+2)(2n+1)}{\rho R^2}\mu\right)\dot{a}_n+\left(\frac{(n+1)(n+2)\sigma}{\rho R^3}+\frac{2(n+2)\mu\dot{R}}{\rho R^3}-\frac{\ddot{R}}{R}\right)(n-1)a_n=0$$

A. Prosperetti, Viscous effects on perturbed spherical flows, Quarterly of Applied Mathematics, 34(4), p. 339-352 (1977).

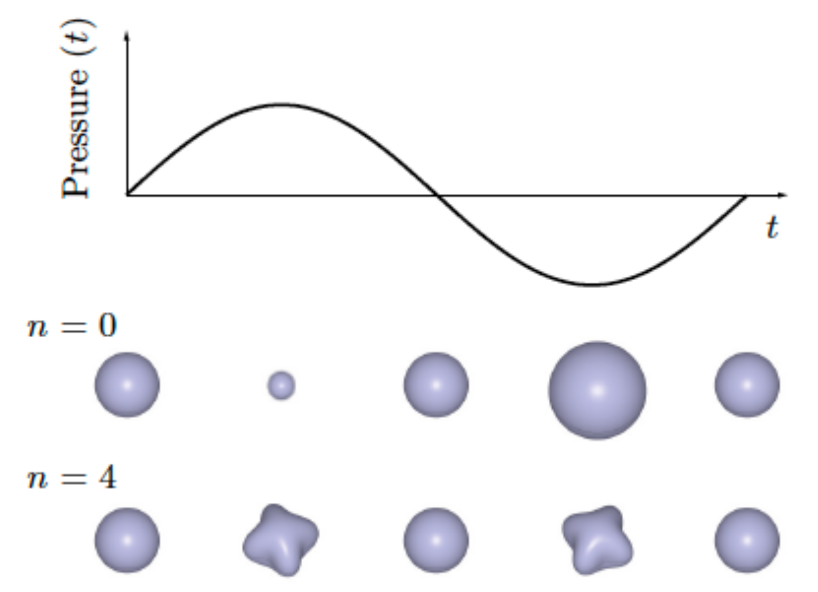


FIG. 1. A bubble oscillating volumetrically within an acoustic field. For positive and negative pressures, compression and expansion phases are observed respectively.

where μ is the viscosity and σ the surface tension. Defining $b(t)=a(t)R^{3/2}$ the equation can be simplified as[19]

$$\ddot{b}_n+\left(\frac{(n-1)(n+1)(n+2)\sigma}{\rho R^3}-\frac{3\dot{R}^2}{4R^2}-\frac{(2n+1)\ddot{R}}{2R}\right)b_n=0. \quad (7)$$

Within this approach, each mode n obeys to the equation of a harmonic oscillator, with time dependent coefficients. The resonance frequencies of the surface modes readily follow from Eq. (7) by considering the static condition $R=R_0$, and $\dot{R}=\ddot{R}=0$.

$$\omega_n=\sqrt{\frac{(n-1)(n+1)(n+2)\sigma}{\rho R_0^3}}, \quad (8)$$

which is the Lamb expression for surface modes for a free gas bubble.

The little lamb(da)

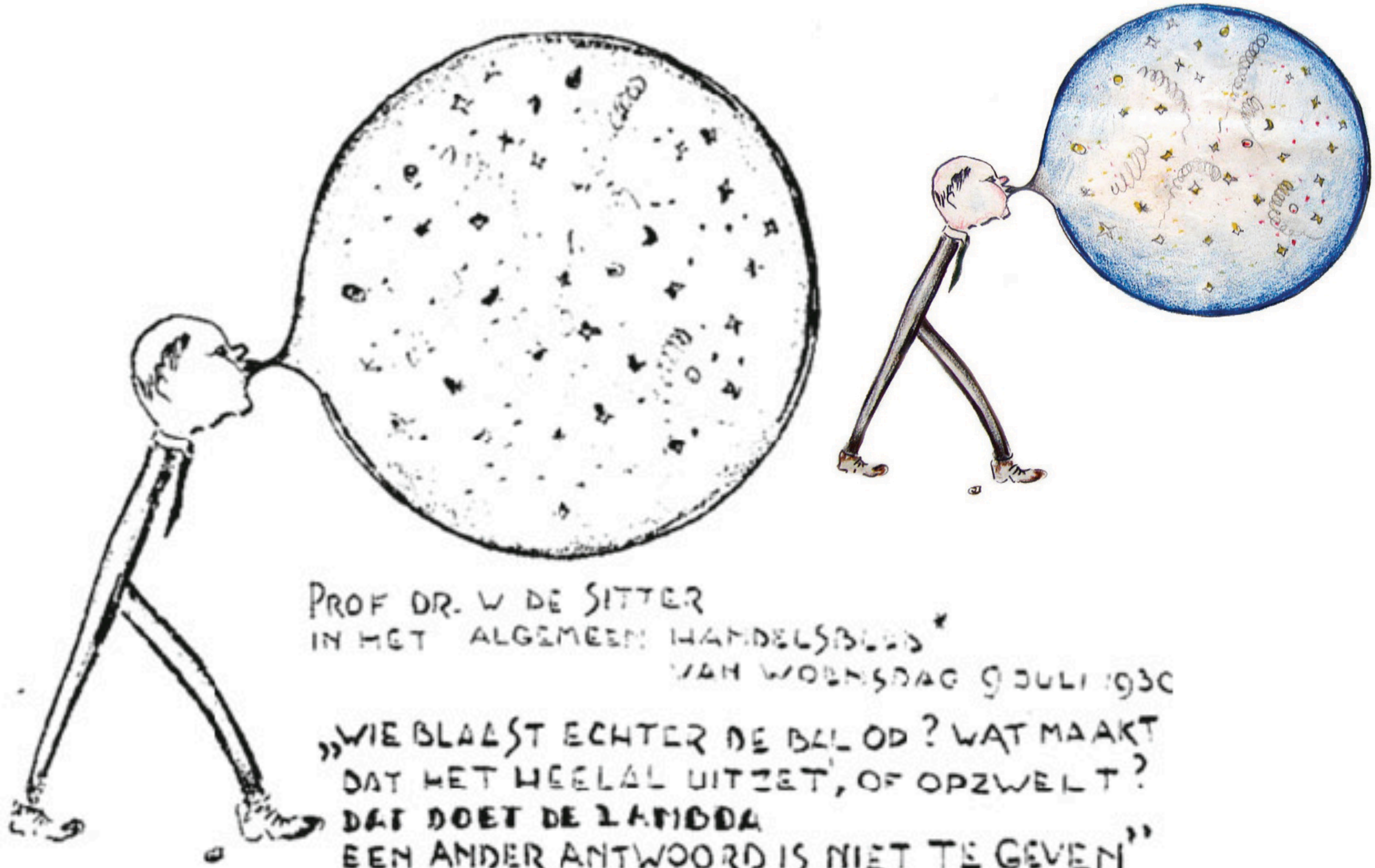


Fig. 3 Sketch of Willem de Sitter (drawn as a "λ") in the *Algemeen Handelsblad* (July 9, 1930), as reproduced in Peebles (1993, p. 81). De Sitter says: "What, however, blows up the ball? What makes the universe expand or swell up? That is done by lambda. No other answer can be given"

The expanding universe. Discussion of Lemaître's solution of the equations of the inertial field, by *W. de Sitter.*

1. *The differential equations.*

In *B. A. N.* 185 it was pointed out that neither of the two possible static solutions of the differential equations

$$(1) \qquad G_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} G + \lambda g_{\mu\nu} + \varkappa T_{\mu\nu} = 0$$

can represent the observed facts of a finite density of matter in space and a systematic velocity of recession of the extragalactic nebulae proportional to the distance, and mention was made of the non-static solution found by Dr. LEMAÎTRE, which is compatible with these two observed facts. In the present article I will discuss some of the consequences of this solution, and will begin by recapitulating it in a notation slightly different from LEMAÎTRE's own. [1])

The conditions of perfect spherical symmetry (isotropy) and perfect homogeneity require the three-dimensional space to be of constant curvature, the three-dimensional line-element thus being

$$(2) \qquad R^2 d\sigma^2 = R^2 \left[d\chi^2 + \sin^2\chi \left(d\psi^2 + \sin^2\psi \, d\theta^2\right)\right].$$

Further the material energy tensor is assumed to be

$$T_{ij} = -g_{ij} p, \quad T_{i4} = 0, \quad T_{44} = g_{44} \rho,$$

where $\rho = \rho_0 + 3p$ is the "relative" density, ρ_0 being the material, or "invariant" density, and p is the pressure, made up of the "kinematical" pressure corresponding to the random motions of the particles, or "molecules", of which the matter is conceived to consist, and the radiation pressure corresponding to the energy of radiation which may be present.

The four-dimensional line-element then is

$$ds^2 = -R^2 d\sigma^2 + f dt^2.$$

In LEMAÎTRE's solution R and f are taken to be functions of t alone. Since we can always put $f dt^2 = c^2 d\tau^2$ and use τ as a new independent variable, we may take $f = c^2$, c being the velocity of light.

The equations (1) then become, if we denote differential quotients d/cdt by dots,

$$(3) \qquad \begin{aligned} 2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{1}{R^2} &= \lambda - \varkappa p \\ \frac{\dot{R}^2}{R^2} + \frac{1}{R^2} &= \frac{1}{3}(\lambda + \varkappa\rho) \end{aligned}$$

and the equation of energy is

$$(4) \qquad \dot{\rho} + 3\frac{\dot{R}}{R}(\rho + p) = 0.$$

LEMAÎTRE puts

$$(5) \qquad \varkappa\rho_0 = \frac{\alpha}{R^3}, \qquad \varkappa p = \frac{\beta}{R^4}.$$

The equation (4) can then be written

$$(4') \qquad R\dot{\alpha} + 3\dot{\beta} = 0.$$

The three equations (3) and (4), or (3) and (4′) are not independent of each other: (4′) can easily be derived from (3). We will use the second of (3) and (4′). An assumption regarding α, or β, must evidently be added in order to make a complete solution possible. The total volume of space is $\pi^2 R^3$, consequently the total mass is, by (5), $\pi^2 \alpha/\varkappa$. This LEMAÎTRE takes to be constant, and consequently by (4′) β is also constant. LEMAÎTRE takes $\beta = 0$.

If we put

$$(6) \qquad y = R\sqrt{\lambda}, \qquad A + \delta = \alpha . \sqrt{\lambda}, \qquad \varepsilon = 3\beta . \lambda,$$

A being a constant number, which must evidently be positive, then y, δ, ε are pure numbers, independent of the choice of units. The equations then become

$$(7) \qquad \begin{aligned} \dot{y}^2 &= \frac{1}{3}\lambda . \frac{y^4 - 3y^2 + Ay + y\delta + \varepsilon}{y^2} \\ y\dot{\delta} + \dot{\varepsilon} &= 0. \end{aligned}$$

It will be shown in article 5 that there is observational

[1]) A full discussion is also contained in a paper by Professor EDDINGTON on the instability of Einstein's spherical world, which is to appear in the May number of the *Monthly Notices of the R. A. S.*

SOVIET PHYSICS USPEKHI (Russian Vol. 81, Nos. 3-4) JANUARY-FEBRUARY 1964

In Memory of A. A. Fridman

ALEKSANDR ALEKSANDROVICH FRIDMAN

(On the seventy-fifth anniversary of his birth)

P. Ya. POLUBARINOVA-KOCHINA

Usp. Fiz. Nauk 80, 345-352 (July, 1963)

Fridman's investigations covered a very wide range. From the beginning of his scientific career he was interested in the basic questions of theoretical geophysics, such as the causes of temperature inversions, the theory of atmospheric vortices and wind gusts, the theory of discontinuities in the atmosphere, and the theory of atmospheric turbulence. He was also ready to tackle any question that had an immediate practical application. Thus, early in his career, he drew up instructions for flying kites carrying meteorographs.

In meteorology, **a baroclinic flow is one in which the density depends on both temperature and pressure. A barotropic flow allows for density dependence only on pressure**, so that the curl of the pressure-gradient force vanishes.

The year 1922 saw the publication of Fridman's book, "Experiment in the Hydromechanics of Compressible Liquids," which lay down the groundwork for theoretical meteorology.

Fridman made the point that in studying atmospheric movements air must be regarded as a compressible baroclinic liquid, i.e., a liquid in which pressure depends not only on density but on the temperature as well. A factor to be taken into account is that the atmosphere receives heat from the sun and loses it through radiation into outer space. Fridman's aim was to pose the basic general problems of the hydrodynamics of compressible liquids.

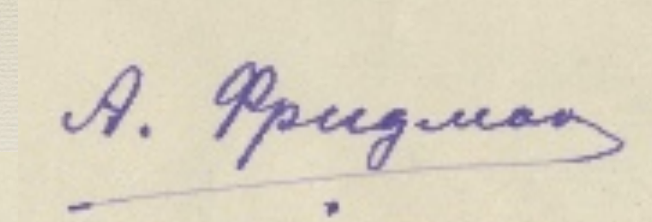
$$p_{baroclinic} = f(\varrho, T)$$

$$p_{barotropic} = f(\varrho)$$

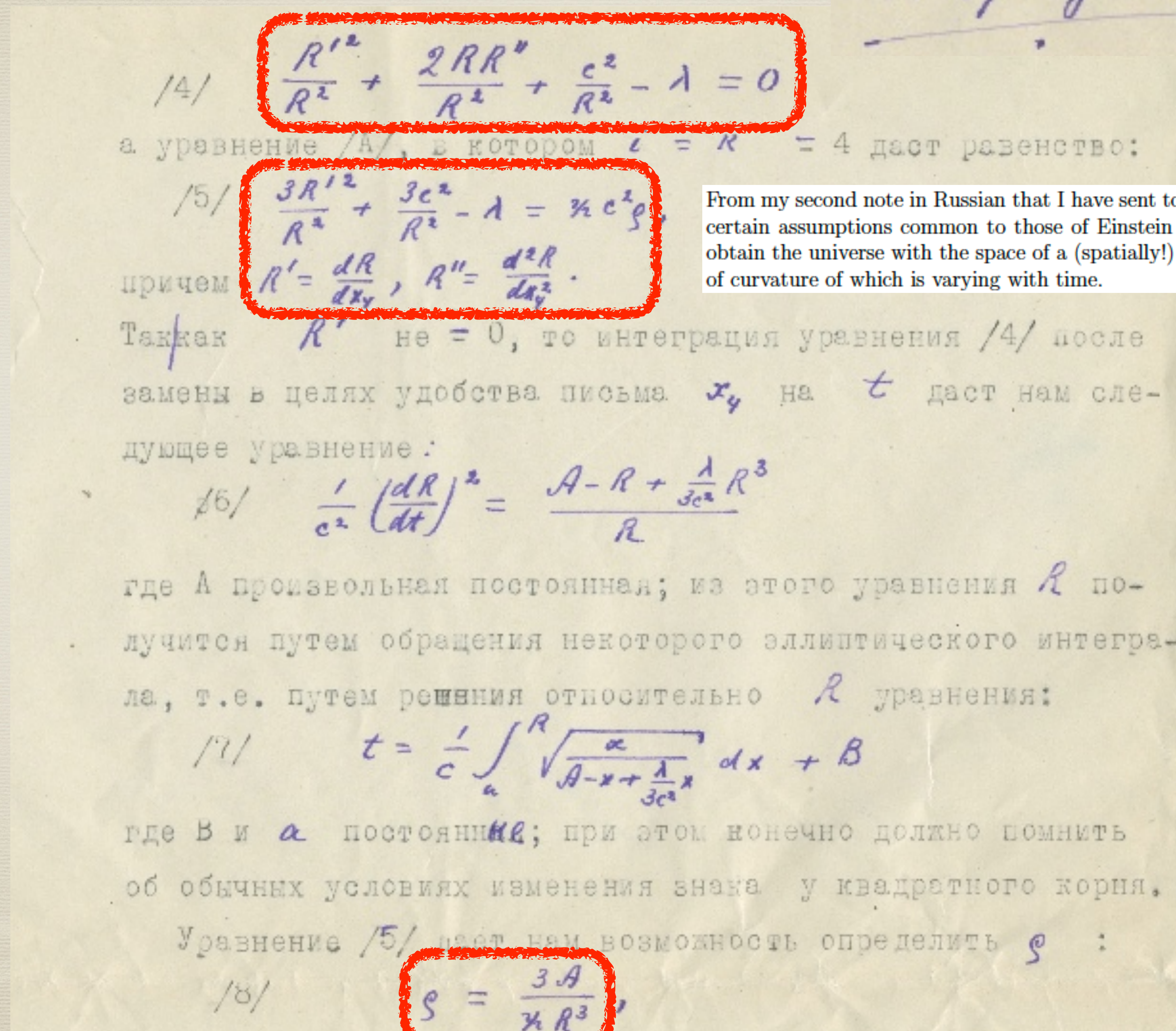
Fridman also distinguishes the case of a compressible liquid in which the total derivative of density with respect to time vanishes and consequently the velocity divergence is zero; he called this phenomenon incompressible motion.

"The waters I step into have never been crossed by anyone."

A. A. Friedman



A. Friedmann, Über die Krümmung des Raumes. Z. Phys., 10, 377-386 (1922). English translation: "On the curvature of space" in Gen. Rel. Grav., 31, 1991-2000 (1999).



/4/ $$\frac{R'^2}{R^2} + \frac{2RR''}{R^2} + \frac{c^2}{R^2} - \lambda = 0$$

а уравнение /A/, в котором $i = k = 4$ даст равенство:

/5/ $$\frac{3R'^2}{R^2} + \frac{3c^2}{R^2} - \lambda = \varkappa c^2 \rho,$$

причем $R' = \frac{dR}{dx_4}$, $R'' = \frac{d^2R}{dx_4^2}$.

Так как R' не $= 0$, то интеграция уравнения /4/ после замены в целях удобства письма x_4 на t даст нам следующее уравнение:

/6/ $$\frac{1}{c^2}\left(\frac{dR}{dt}\right)^2 = \frac{A - R + \frac{\lambda}{3c^2}R^3}{R}$$

где A произвольная постоянная; из этого уравнения R получится путем обращения некоторого эллиптического интеграла, т.е. путем решения относительно R уравнения:

/7/ $$t = \frac{1}{c}\int_a^R \sqrt{\frac{x}{A - x + \frac{\lambda}{3c^2}x^3}}\,dx + B$$

где B и a постоянные; при этом конечно должно помнить об обычных условиях изменения знака у квадратного корня.

Уравнение /5/ дает нам возможность определить ρ:

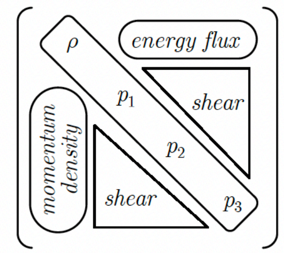
/8/ $$\rho = \frac{3A}{\varkappa R^3},$$

From my second note in Russian that I have sent to you, you have seen that under certain assumptions common to those of Einstein and De Sitter it is possible to obtain the universe with the space of a (spatially!) constant curvature, the radius of curvature of which is varying with time.

A. Friedmann, A. Über die Möglichkeit einer Welt mit konstanter negativer Krümmung des Raumes, Z. Phys., 21, 326-332 (1924). English translation: "On the Possibility of a World with Constant Negative Curvature of Space". Gen. Rel. Grav. 1999, 31, 2001-2008.

https://www.ilorentz.org/history/Friedmann_archive/

$$T^\mu_\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

$T_{\mu\nu} =$ 

$T^{00} = \rho =$ energy density
$T^{0i} =$ energy flux components
$T^{i0} =$ momentum density components
$T^{ii} = p_i =$ pressure components
$T^{ij} =$ shear components ($i \neq j$)
$i, j = 1, 2, 3$

Nous désignons par ρ la densité de l'énergie totale, la densité de l'énergie rayonnante sera $3p$ et la densité de l'énergie concentrée dans la matière est $\delta = \rho - 3p$.

Il faut identifier ρ et $-p$ avec les composantes T_4 et $T_1 = T_2 = T_3$ du tenseur d'énergie matérielle et δ avec T. Calculons les composantes du tenseur de Riemann contracté pour un univers d'intervalle

$$ds^2 = -R^2 d\sigma^2 + dt^2 \tag{1}$$

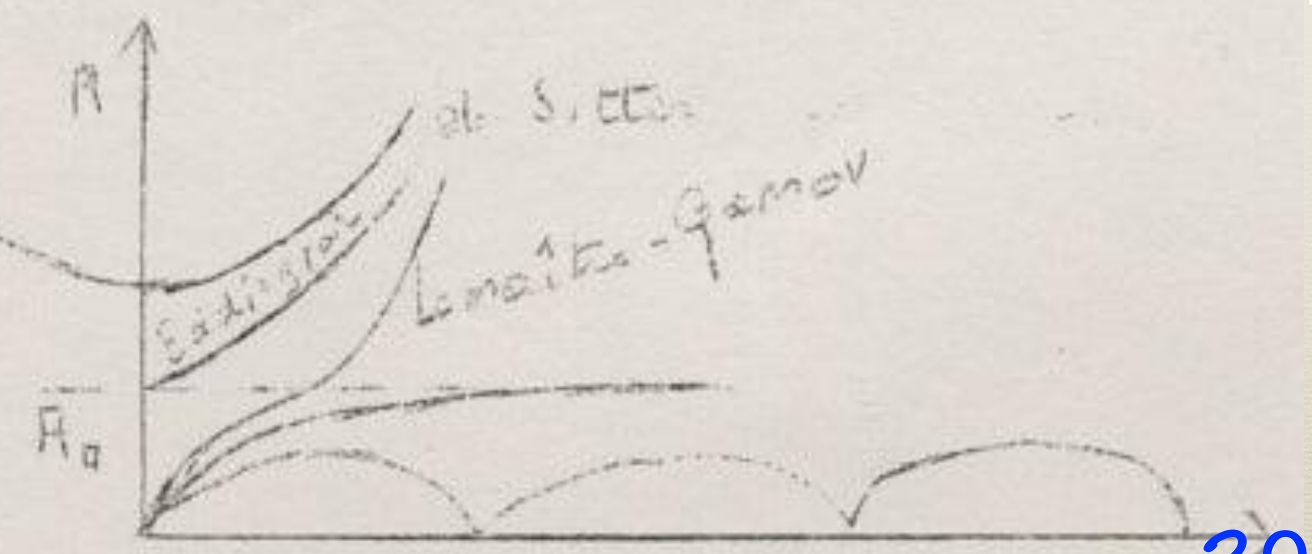
$d\sigma$ est l'élément de longueur d'un espace de rayon égal à un ; le rayon R de l'espace est une fonction du temps. Les équations du champ de gravitation s'écrivent

$$3\frac{R'^2}{R^2} + \frac{3}{R^2} = \lambda + \kappa\rho \quad (I) \tag{2}$$

et

$$2\frac{R''}{R} + \frac{R'^2}{R^2} + \frac{1}{R^2} = \lambda - \kappa p \quad (II) \tag{3}$$

G. Lemaître, The expanding universe, Mon. Not. R. Astron. Soc., 41, 491-501 (1931).



https://archives.uclouvain.be/atom/index.php/archives-de-georges-lemaitre

G. Lemaître, Un univers homogène de masse constante et de rayon croissant, rendant compte de la vitesse radiale des nébuleuses extra-galactiques. Annales de la Société Scientifique de Bruxelles, 47A, 49-59 (1927). English translation: "A homogeous universe with constant mass and increasing radius accounting for the radial velocities of the radial velocities of the extra-galactic nebulae," Gen. Rel. Grav., 45, 1635-1646 (2013).

Cherchons une solution pour laquelle la masse totale $M = V\delta$ demeure constante. Nous pourrons alors poser

$$\kappa\delta = \frac{\alpha}{R^3} \tag{5}$$

misprint

où α est une constante. Tenant compte de la relation

$$\rho = \delta + 3p$$

existant entre les diverses sortes d'énergie, le principe de conservation de l'énergie devient

$$3\,d(pR^3) + 3p\,R^2\,dR = 0 \tag{7}$$

dont l'intégration est immédiate ; β désignant une constante d'intégration, nous avons

$$\kappa p = \frac{\beta}{R^4} \tag{8}$$

et donc

$$\kappa\rho = \frac{\alpha}{R^3} + \frac{3\beta}{R^4} \tag{9}$$

Substituant dans (2), nous avons à intégrer

$$\frac{R'^2}{R^2} = \frac{\lambda}{3} - \frac{1}{R^2} + \frac{\kappa\rho}{3} = \frac{\lambda}{3} - \frac{1}{R^2} + \frac{\alpha}{3R^3} + \frac{\beta}{R^4} \tag{10}$$

ou

$$t = \int \frac{dR}{\sqrt{\frac{\lambda R^2}{3} - 1 + \frac{\alpha}{3R} + \frac{\beta}{R^2}}} \tag{11}$$

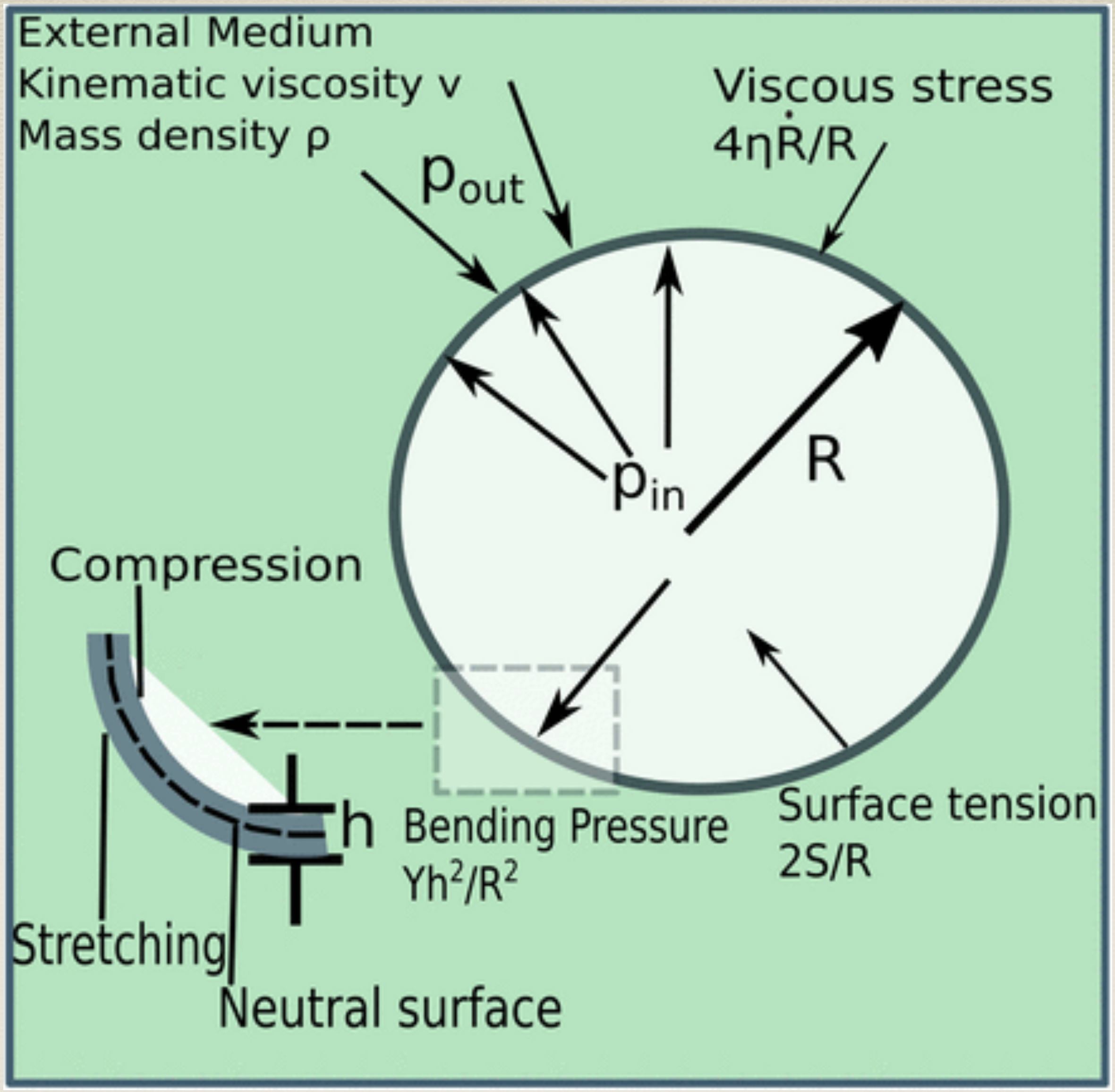
This was also Albert Einstein's original feeling. On 13 April 1917 he wrote to Willem de Sitter,[17]

> In any case, one thing stands. The general theory of relativity *allows* the addition of the term $\lambda g_{\mu\nu}$ in the field equations. One day, our actual knowledge of the composition of the fixed star sky, the apparent motions of fixed stars, and the position of spectral lines as a function of distance, will probably have come far enough for us to be able to decide empirically the question of whether or not λ vanishes. Conviction is a good motive, but a bad judge.

Visco-Elastic Rayleigh-Plesset Equation

$$\omega = \frac{2i\nu}{R_0^2} \mp \frac{1}{2R_0^2}\sqrt{-16\nu^2 - 4R_0^2\left(\frac{\Delta P}{\rho} - \frac{Yh^2}{\rho R_0^2}\right)}.$$

$$\omega_0 = \sqrt{\Delta P/\rho R_0^2}$$



Hypothesis:

Incompressible flow of a compressible liquid (cf. Friedmann's 1922 PhD Thesis).

$$\frac{p_{in}(t) - p_{out}(t)}{\rho} + \frac{Yh^2}{\rho R^2} = R\frac{d^2R}{dt^2} + \frac{3}{2}\left(\frac{dR}{dt}\right)^2 + \frac{4\nu}{R}\frac{dR}{dt} + \frac{2S}{\rho R}$$

A. N. Malmi-Kakkada & D. Thirumalai, Generalized Rayleigh-Plesset Theory for Cell Size Maintenance in Viruses and Bacteria, bioRxiv preprint 552778 (2019).

Manipulating the Friedman-Lemaître Equations

$$\tfrac{1}{3}(I) \qquad \left(\frac{\dot{a}}{a}\right)^2 + \kappa c^2 \left(\frac{1}{a}\right)^2 = \frac{8\pi G}{3}\varrho + \frac{\Lambda c^2}{3} \qquad \lambda = \Lambda c^2$$

$$\tfrac{1}{2}(II) \qquad \frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 + \frac{1}{2}\kappa c^2 \left(\frac{1}{a}\right)^2 = -\frac{8\pi G}{3}\frac{3p}{2c^2} + \frac{\Lambda c^2}{2}$$

$$\tfrac{1}{2}(II) - \tfrac{1}{2}\tfrac{1}{3}(I) \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\varrho + 3\frac{p}{c^2}\right) + \frac{\Lambda c^2}{3}$$

$$p_{barotropic} = f(\varrho)$$

$$(III) \quad p = w\varrho c^2$$

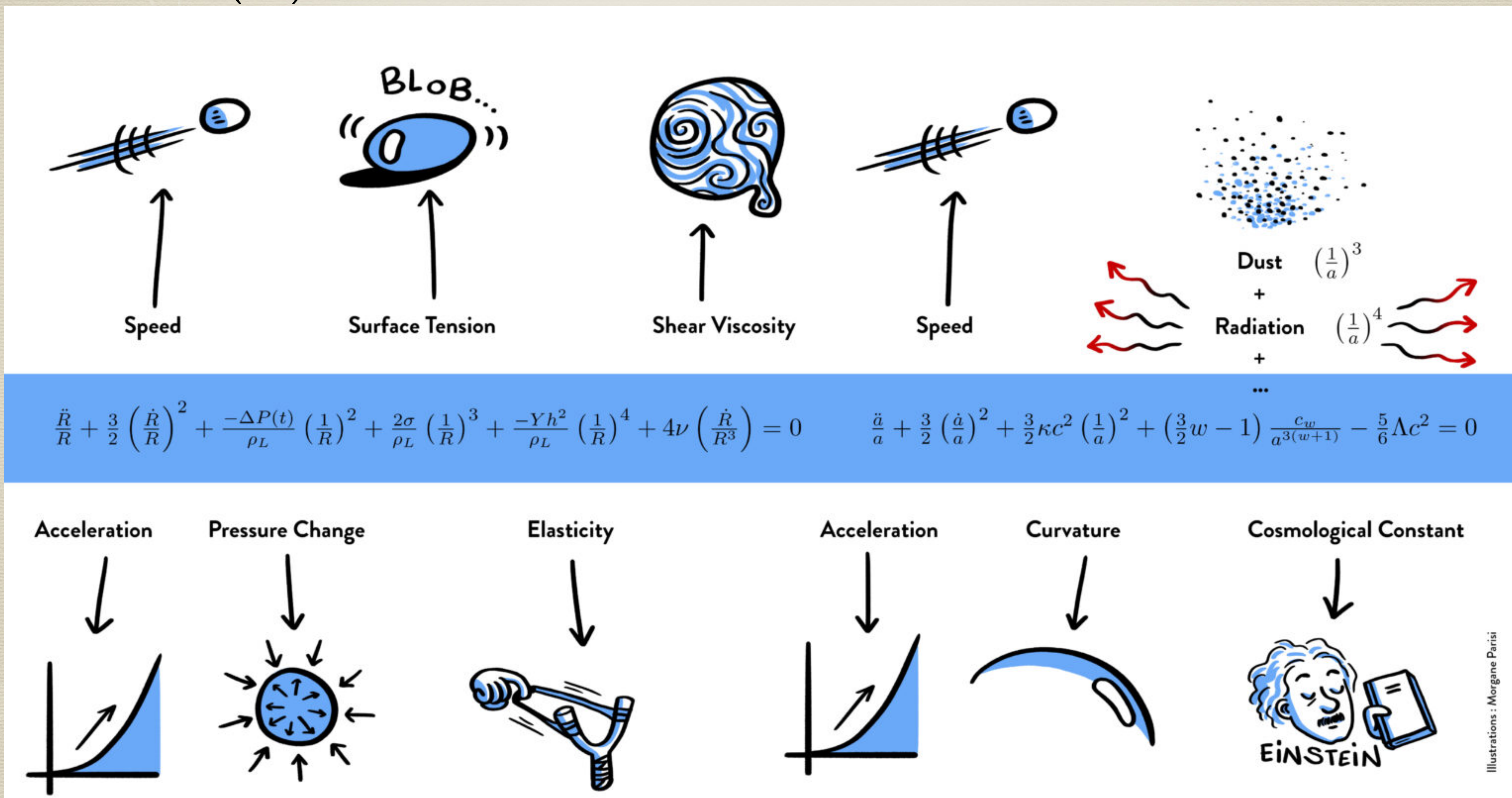
$$\tfrac{1}{2}(II) + \tfrac{1}{3}(I) \quad \frac{\ddot{a}}{a} + \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2 + \frac{3}{2}\kappa c^2 \left(\frac{1}{a}\right)^2 = \frac{4\pi G}{3}(2 - 3w)\varrho + \frac{5}{6}\Lambda c^2$$

$$(IV) \qquad \dot{\varrho} + 3\frac{\dot{a}}{a}\left(\varrho + \frac{p}{c^2}\right) = 0 \qquad \varrho = \sum_i \frac{3}{8\pi G}\frac{c_{w_i}}{a^{3(1+w_i)}}$$

$$\frac{\ddot{a}}{a} + \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2 + \frac{3}{2}\kappa c^2 \left(\frac{1}{a}\right)^2 = \sum_i (1 - \frac{3w_i}{2})\frac{c_{w_i}}{a^{3(1+w_i)}} + \frac{5}{6}\Lambda c^2$$

$$\frac{\ddot{a}}{a} + \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2 + \frac{3}{2}\kappa c^2\left(\frac{1}{a}\right)^2 = \sum_i (1 - \frac{3w_i}{2})\frac{c_{w_i}}{a^{3(1+w_i)}} + \frac{5}{6}\Lambda c^2$$

$$\frac{\ddot{R}}{R} + \frac{3}{2}\left(\frac{\dot{R}}{R}\right)^2 + \frac{-\Delta P}{\rho_L}\left(\frac{1}{R}\right)^2 = -\frac{2\sigma}{\rho_L}\left(\frac{1}{R}\right)^3 + \frac{Yh^2}{\rho_L}\left(\frac{1}{R}\right)^4 - 4\nu\frac{\dot{R}}{R^3}$$



G. Rousseaux & S. C. Mancas, Visco-elastic cosmology for a sparkling universe?, General Relativity and Gravitation, 52(6), 1-8 (2020).

ENTROPY GENERATION AND THE SURVIVAL OF PROTO-GALAXIES IN AN EXPANDING UNIVERSE*

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Received 1971 February 18; revised 1971 April 6

III. ENTROPY PRODUCTION IN ISOTROPIC HOMOGENEOUS COSMOLOGIES

We now consider a universe described by the Robertson-Walker metric:

$$-g_{\mu\nu}dx^\mu dx^\nu = dt^2 - R^2(t)[dr^2/(1-kr^2) + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2] . \tag{3.1}$$

This is written in a comoving coordinate system, so the velocity is

$$U^r = U^\theta = U^\phi = 0 ; \quad U^t = 1 . \tag{3.2}$$

It is straightforward to calculate the energy-momentum tensor (2.60) here as

$$T_{ij} = (p - 3\zeta\dot{R}/R)g_{ij} \tag{3.3}$$

$$T_{it} = 0 , \tag{3.4}$$

$$T_{tt} = \rho , \tag{3.5}$$

where now i, j, etc., run over the labels r, θ, ϕ. Thus, as far as the Einstein field equations are concerned, the only effect of dissipation is to replace p with $p - 3\zeta\dot{R}/R$. In particular, we still get precisely the same equation for $\dot{R}$ in terms of ρ as in the adiabatic case:

$$\dot{R}^2 + k = \tfrac{8}{3}\pi\rho G R^2 . \tag{3.6}$$

However, the bulk viscosity does appear in the conservation equation (2.63), which here reads

$$\frac{d}{dt}(\rho R^3) = -3R^2\dot{R}(p - 3\zeta\dot{R}/R) \tag{3.7}$$

The particle conservation equation (2.64) is unaffected by dissipation:

$$\frac{d}{dt}(nR^3) = 0 . \tag{3.8}$$

The Astrophysical Journal, 168:175–194, 1971 September 1

Bubble Puzzles

Bubbles are familiar from daily life and occupy an important role in physics, chemistry, medicine, and technology. Nevertheless, their behavior is often surprising and unexpected—and, in many cases, still not understood.

Detlef Lohse

propeller gets damaged. Rayleigh mathematically described the dynamics[6] of such a collapsing void in water, assumed to be spherical with radius $R(t)$, and laid the foundation of what is now called the Rayleigh–Plesset equation,[3,7]

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho}\left[p_g - P_0 - P(t)\right] - 4\nu\frac{\dot{R}}{R} - \frac{2\sigma}{\rho R}. \qquad (4)$$

Here ν denotes the kinematic viscosity, p_g the gas pressure inside the bubble (dependent on the radius), and $P(t)$ the time-dependent external pressure.

Rayleigh–Plesset dynamics can lead to energy focusing, as can be seen by neglecting all terms on the right-hand side of equation 4, that is, by considering only the inertial terms, $R\ddot{R} + 3/2\,\dot{R}^2 = 0$. Integration immediately gives $R(t) = R_0[(t_* - t)/t_*]^{2/5}$, with the remarkable feature of a divergent bubble-wall velocity as t approaches the time t_* of the bubble collapse. It is this finite-time singularity that leads to the cavitation damage. The collapse is eventually cut off by the adiabatic compression (and thus heating) of the gas inside the bubble and by the sound emission at bubble collapse,[8] or in many cases also by the disintegration of the bubble. The emitted sound pressure (equation 3) obviously also diverges.

ble also will oscillate periodically around the ambient radius R_0 that the bubble would have under static, ambient conditions.

If instead the bubble is kicked with a single pressure pulse, the bubble's resonance frequency f_0 survives longest; all other frequencies damp out earlier. To calculate the resonance frequency, one needs the restoring force, which results from the pressure in the gas bubble. For large enough bubbles, $R_0 \gg \sigma/P_0 \approx 1\ \mu m$, the force depends on the ambient pressure P_0 and the actual radius $R(t)$, and the resonance frequency is given by[3]

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{3\gamma P_0}{\rho R_0^2}}. \qquad (2)$$

Here γ is the adiabatic exponent, the ratio of the constant-pressure and constant-volume heat capacities of the gas. For air bubbles (for which $\gamma = 1.4$) in water under standard conditions, equation 2 reduces to $f_0 R_0 \approx 3$ kHz mm.

Most remarkably, this eigenfrequency of bubbles can be heard as the underwater sound of rain. When raindrops fall on a calm lake, the underwater sound is *not* generated at drop impact. Rather, at impact, a small air bubble is entrained, as shown in figure 1. Due to the violent entrainment process, the bubble experiences a pressure kick and subsequently oscillates at its eigenfrequency.[4] We hear the corresponding sound emission from the resulting pressure field

$$P_s(r,t') = \frac{\rho R}{r}\left(2\dot{R}^2 + R\ddot{R}\right) \qquad (3)$$

at large distances r from the bubble at the delayed time $t' = t + r/c$, where c is the speed of sound in water. Typically, the entrained bubble has a radius of about 0.2 mm, corresponding to a resonance frequency around $f =$ 15 kHz, which is in the audible range. If the raindrop is too small or too large, no bubble is entrained and the sound is shut off. Correspondingly, surfactants can suppress air entrainment and the sound of rain.[4]

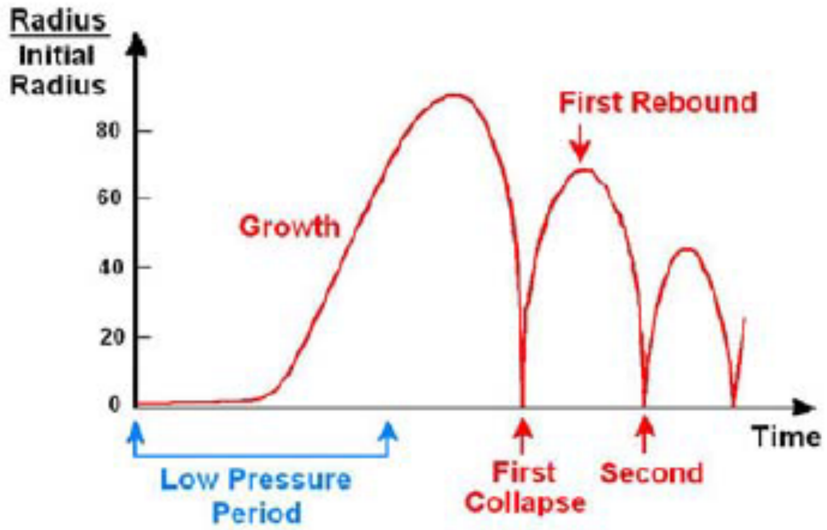


Figure 3. Typical bubble radius response to an episode of low pressure according to the Rayleigh-Plesset equation.

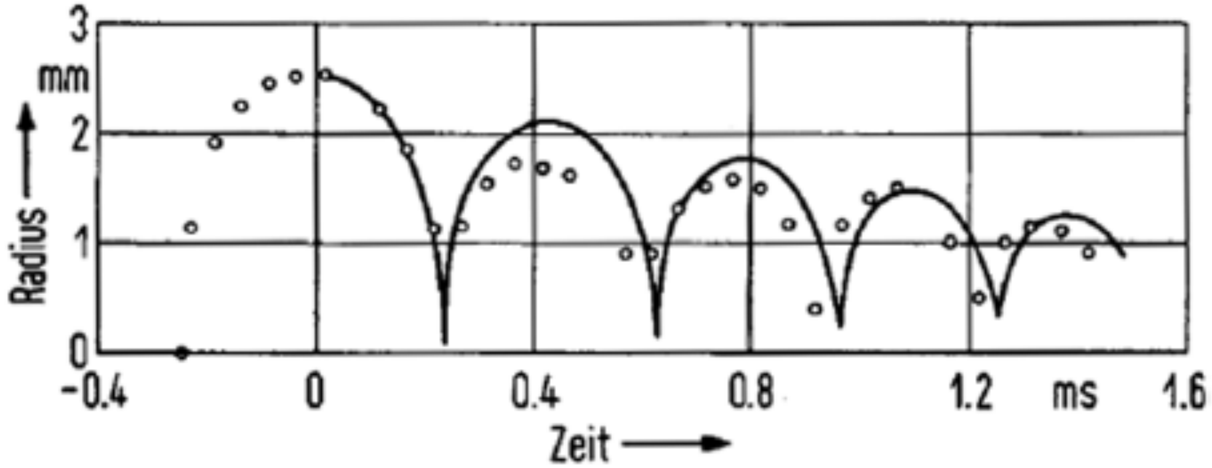


Figure 32. Decaying bubble oscillation in silicone oil (open circles). Comparison with the Rayleigh–Plesset model (solid line). Viscosity $\mu = 0.485$ Pa s.

Physics of bubble oscillations

Werner Lauterborn and Thomas Kurz

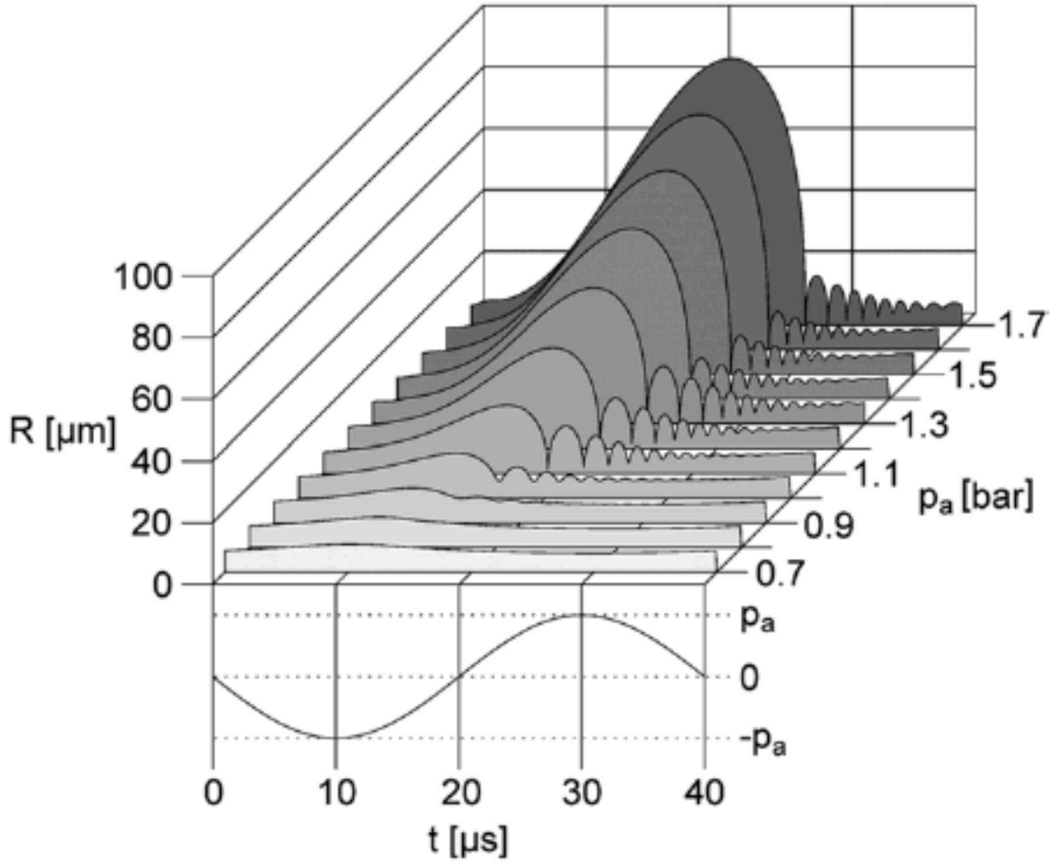


Figure 4. Sequence of calculated steady state bubble oscillations for different acoustic pressure amplitudes p_a in the giant response region (Gilmore model). Bubble radius at rest $R_n = 7\,\mu$m, driving frequency $\nu_a = 25$ kHz. (Courtesy of R Geisler.)

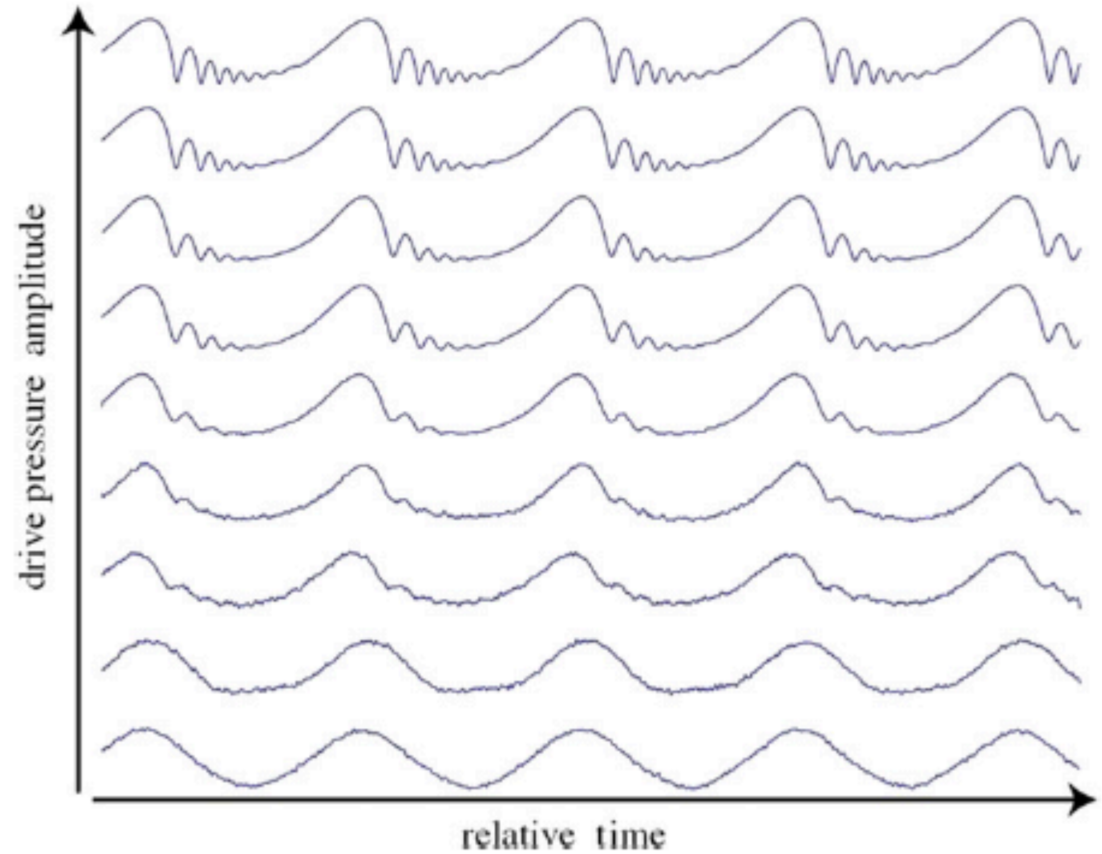
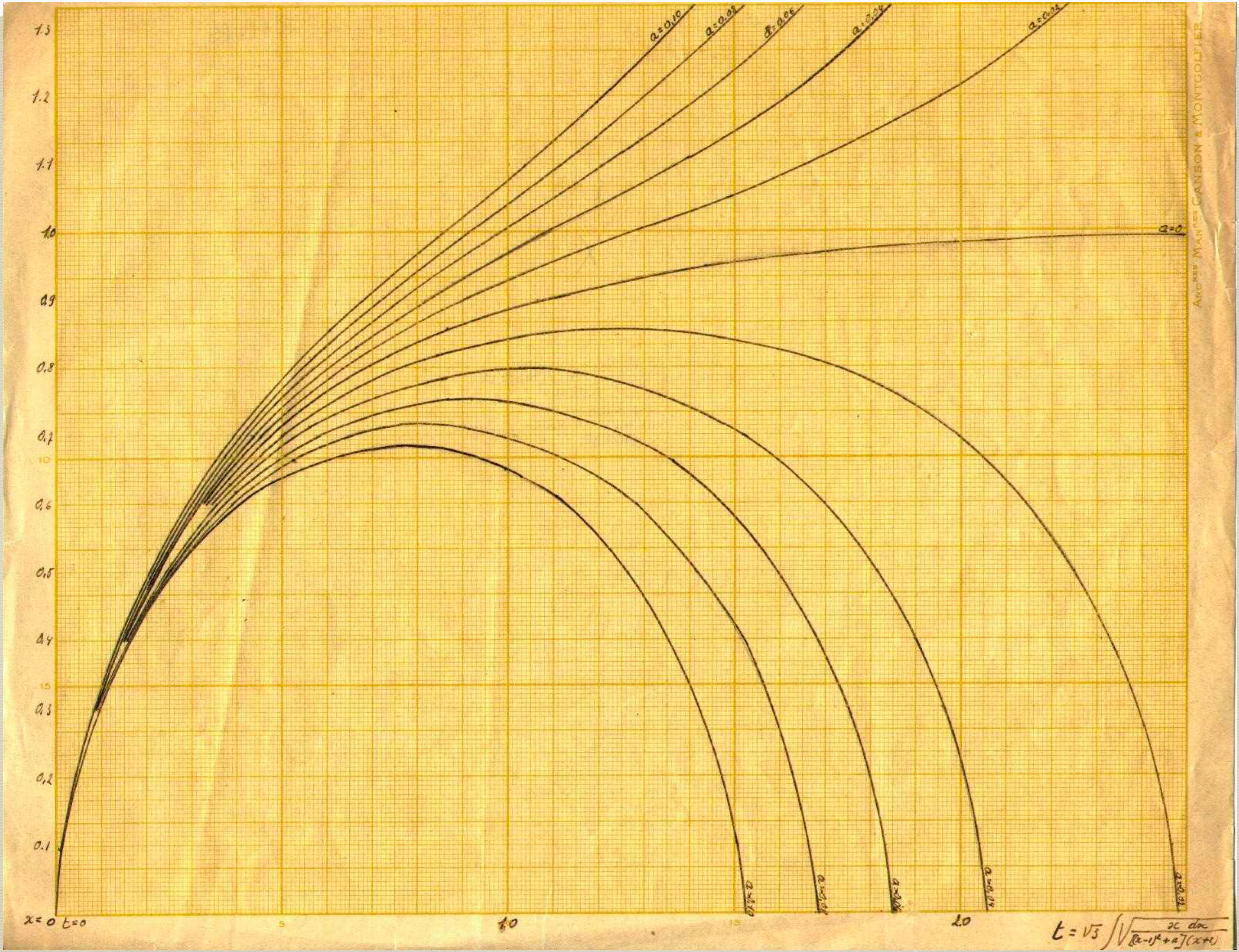
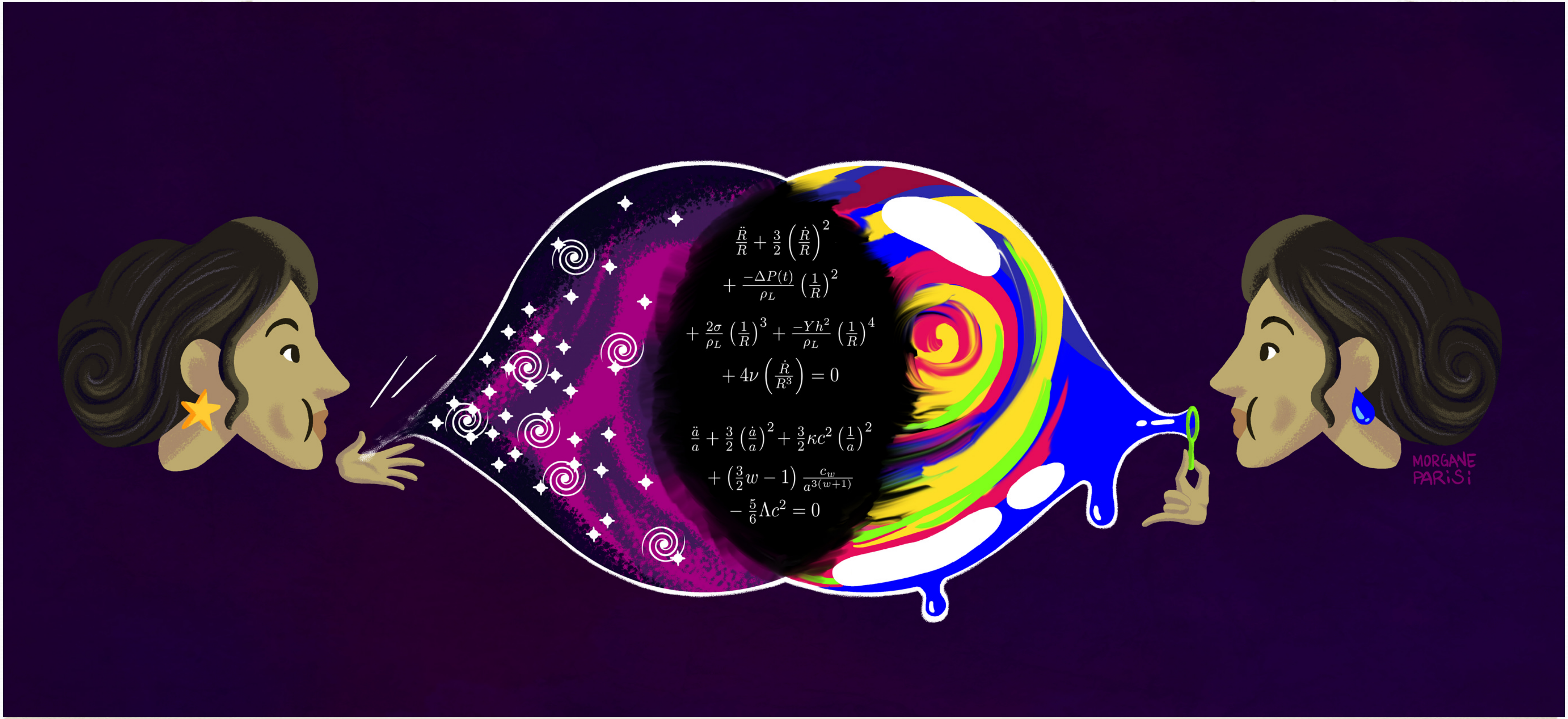


Figure 25. Sequence of measured steady state bubble oscillations for different acoustic pressure amplitudes p_a. Driving frequency $\nu_a = 25$ kHz. (Reprinted with permission from T J Matula [125]. Copyright 1999 by The Royal Society.)



G. Lemaître, unpublished, red notebook circa 1927.

https://archives.uclouvain.be/atom/index.php/archives-de-georges-lemaitre



"Those solutions where the universe expands and contracts successively while periodically reducing itself to an atomic mass of the dimensions of the solar system, have an indisputable charm and make one think of the Phoenix of legend"