LOOKING FOR THE BLACK HOLE LASER EFFECT IN INTERFACIAL HYDRODYNAMICS

Alexis BOSSARD (Pprime Institute,Poitiers, France)

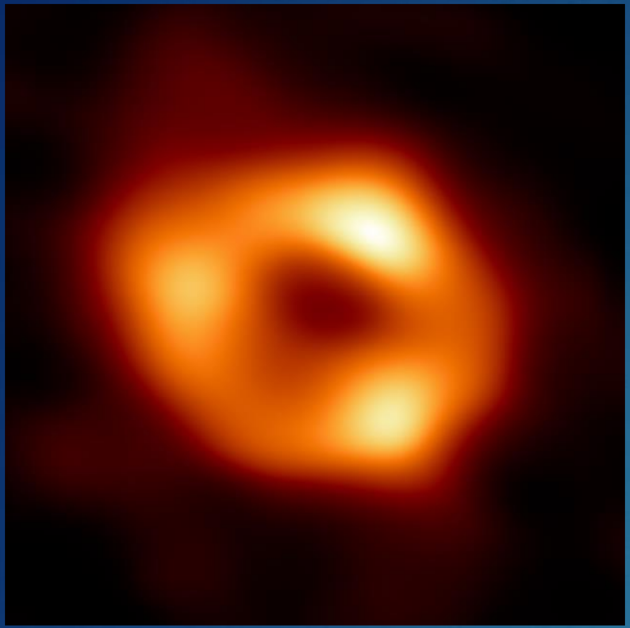
2nd year of PhD thesis under the supervision of Germain Rousseaux (CNRS)

08 november 2023, CGR COPHY « Transverse task forces »

Cosmological Physics GDR

Analogue gravity in interfacial hydrodynamics

William Unruh (1981)[1]



Source: https://Beta.NSF.GOV/EHT

$$\mathbf{Fr}_{\mathbf{local}} = \mathbf{1} \Rightarrow \text{Existence of an analogous horizon}$$

Weinfurtner et al (2011)[4]

- $Fr = \frac{U(x)}{c} = \frac{U(x)}{\sqrt{gh}}$ with $g = 9.81\ m.s^{-2}$ and h the water depth
- $ds^2 = -c^2\left(1 - \frac{U^2}{c^2}\right)dt^2 + 2U dt dx + dx^2$

U = Solution of Navier-Stokes equations

- $(\omega - Uk)^2 = c^2k^2$ with no dispersive terms

With dispersive terms: $(\omega - Uk)^2 = gk\left(1 + \frac{\gamma}{\rho g}k^2\right)\mathrm{th}\,(kh)$

- $T = \frac{\hbar}{4\pi k_B}\left|\frac{1}{c}\frac{\partial\left(c^2 - U^2\right)}{\partial x}\right|\Bigg)_{x=Fr^{-1}(1)}$

Stephen Hawking (1974)[2]

Matt Visser (1998)[3]

White hole: time reversal of the black hole($t \rightarrow -t$) ~ White fountain

Euvé et al (2016) [5]

Black hole lasers

Steven Corley*
Theoretical Physics Institute, Department of Physics, University of Alberta, Edmonton, Alberta, Canada T6G 2J1

Ted Jacobson†
Department of Physics, University of Maryland, College Park, MD 20742-4111, USA

Vilenkin (1978) [6] : one active semi-transparent mirror + one mirror

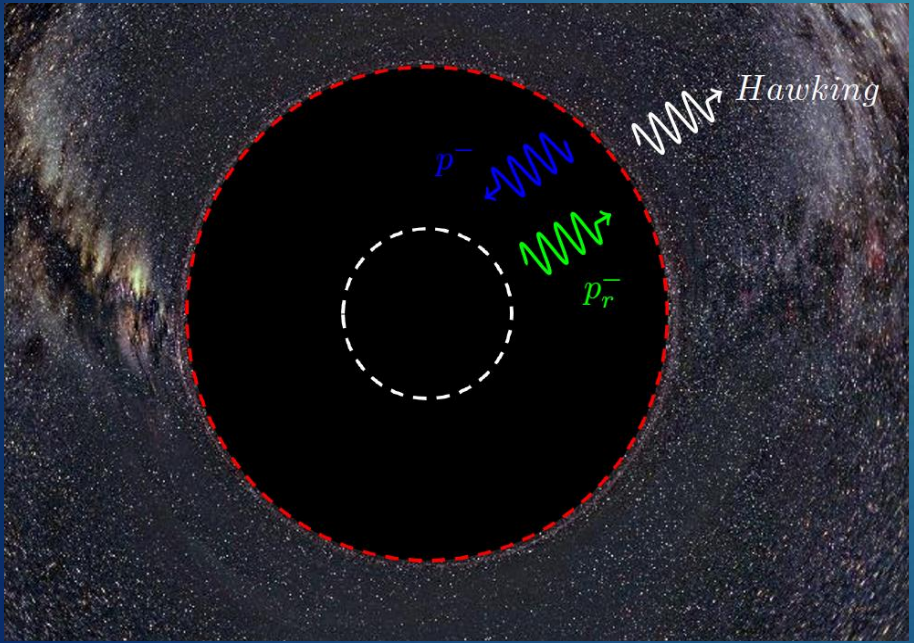
Corley and Jacobson (1999) [7]: two active semi-transparent mirrors

Definition of the black hole laser effect: it is the amplification of Hawking radiation due to successive bouncing of trapped modes on two horizons which would act as active mirrors as in an optical laser

$$(\omega - vk)^2 = c^2 F(k)^2$$

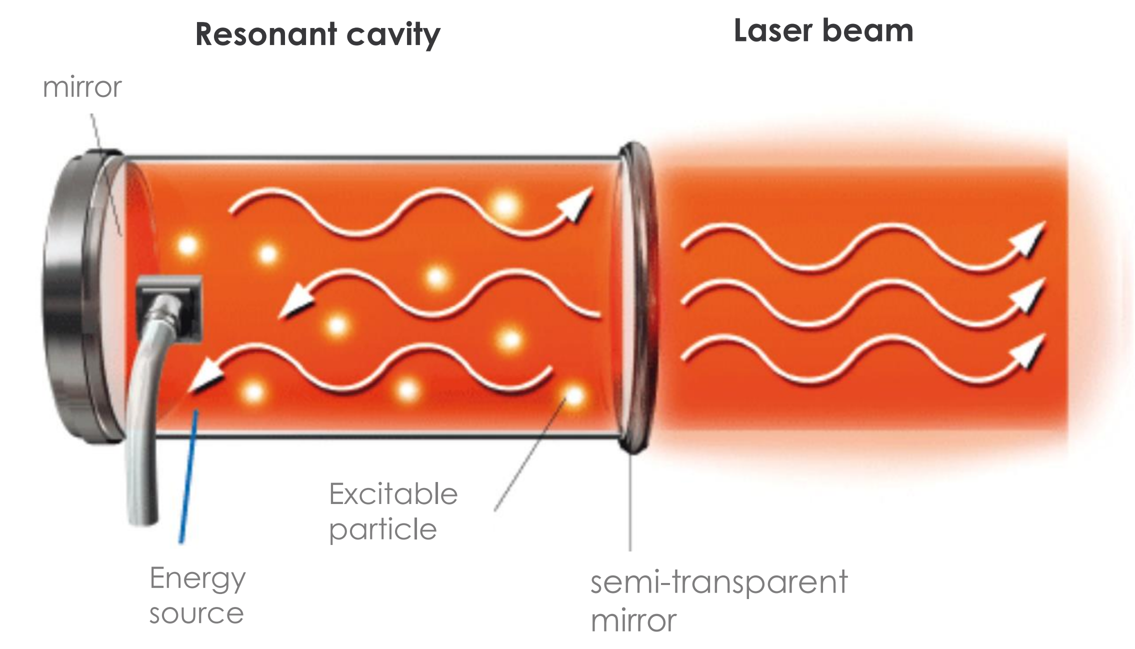
$$F(k)^2 = k^2 \pm \frac{1}{k_0^2} k^4$$

Superluminal correction in supercritical region	Examples: BEC, circular jump
Subluminal correction in subcritical region	Example: Flow in a free surface channel

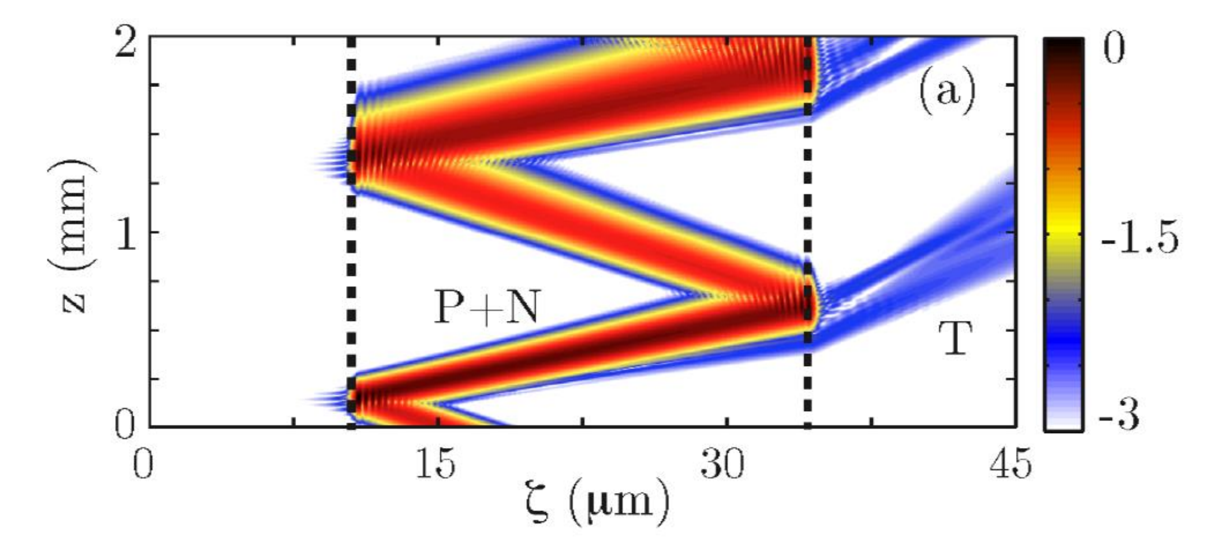


schematic representation of a laser effect between an inner and outer horizon

~ analogous to… (kinematically)



Numerical existence of the black hole laser effect



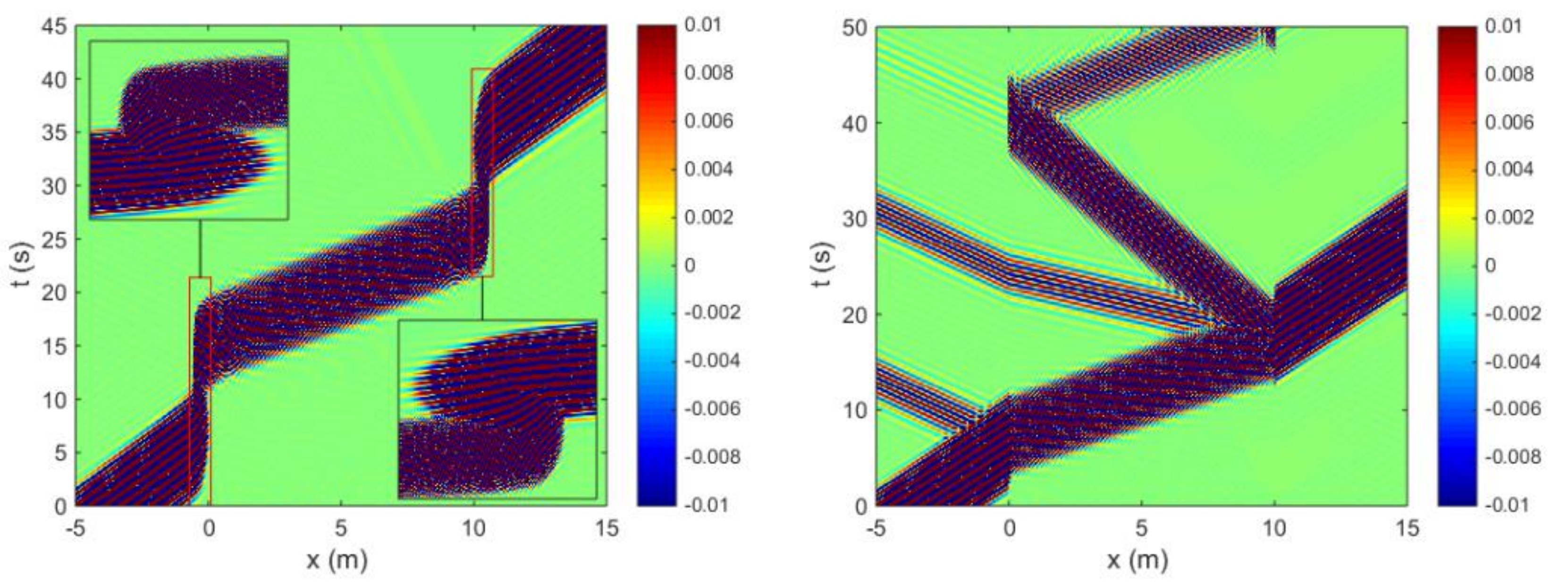
Faccio et al (2012) [8]: Optical black hole laser

Without dissipation

Robertson and Rousseaux [10]

Peloquin et al (2016) [9]:
Hydrodynamic black hole laser

- The velocity field is imposed
- Stable horizons



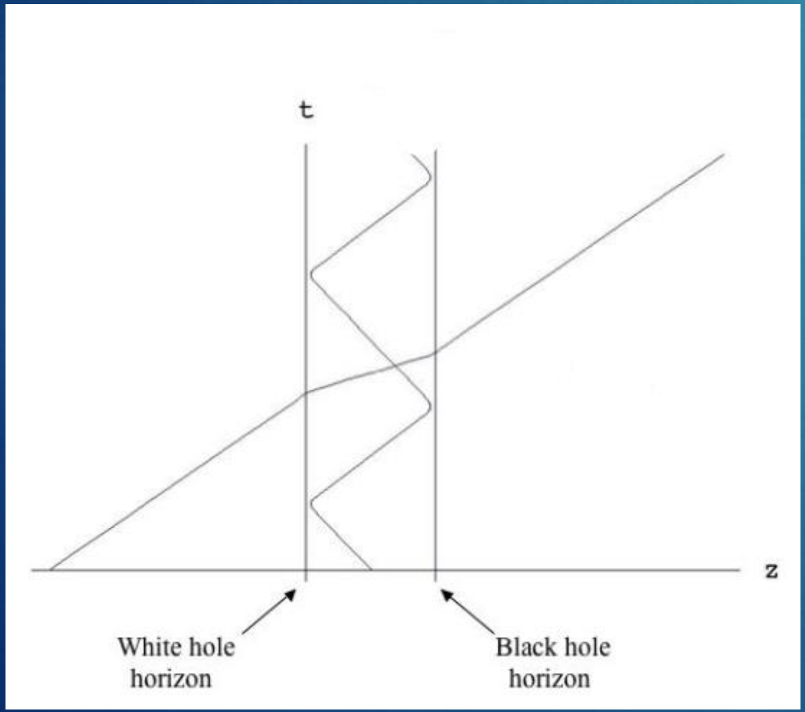
Low speeds gradients

High speeds gradients

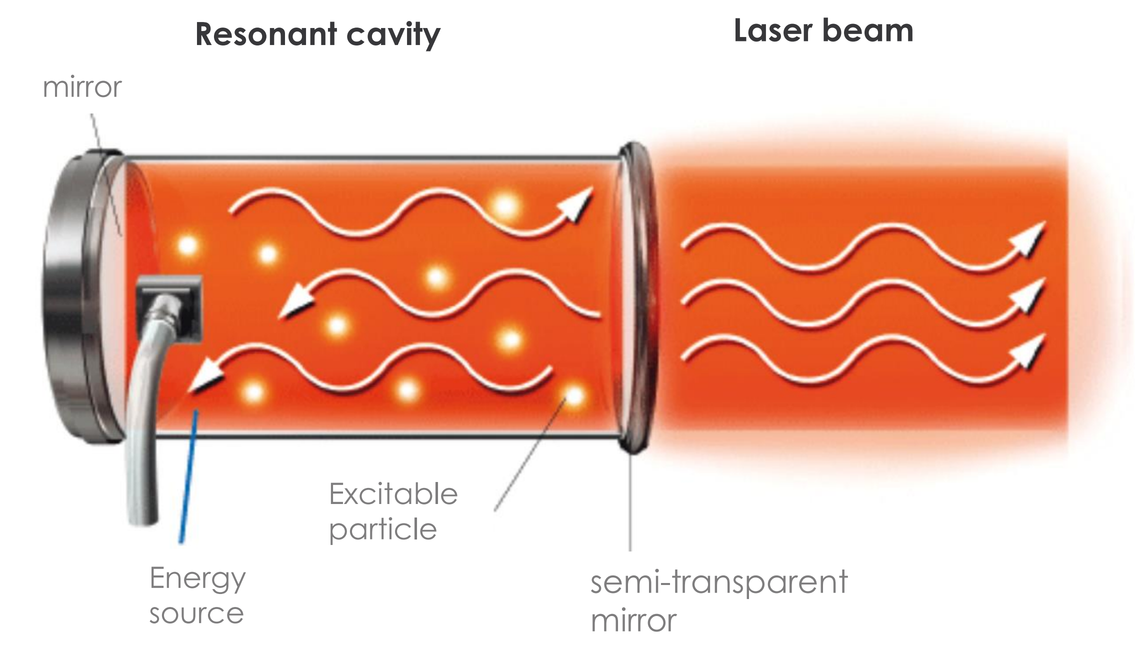
Ingredients for the usual black laser effect (as discussed in literature):

1. You need two stable horizons
2. A superluminal or subluminal dispersive correction
3. A trapping cavity with a flow regime compatible with the dispersive regime
4. Mixing of positive and negative modes
5. To avoid modes dissipation

No experimental measurement Steinhauer (2022) [11]



~ analogous to… (kinematically)



I) Flow over 2 obstacles :

Pratt (1984) [12]

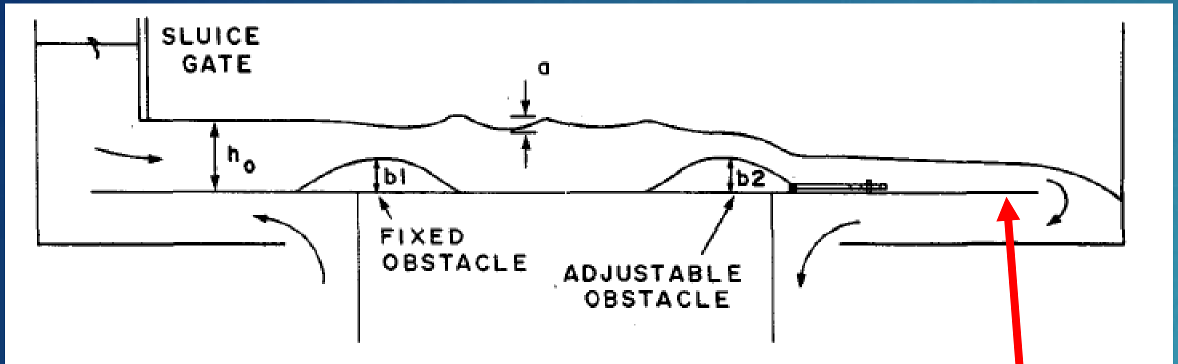


FIG. 3. Cross-section of laboratory flume.

Distance between obstacles:
4*length of the first obstacle

Channel too short to observe the influence of downstream dissipation

Definitions (from local Fr) :

- Transcritical flow:
Transition from Fr<1 to Fr>1 (or vice versa)
- Supercritical flow: Fr>1
- Subcritical flow: Fr<1

- Froude number

$$Fr = \frac{U}{c} = \frac{U}{\sqrt{gh}}$$

- Pratt number

$$\mathcal{P} = \frac{b_2 - b_1}{b_1}$$

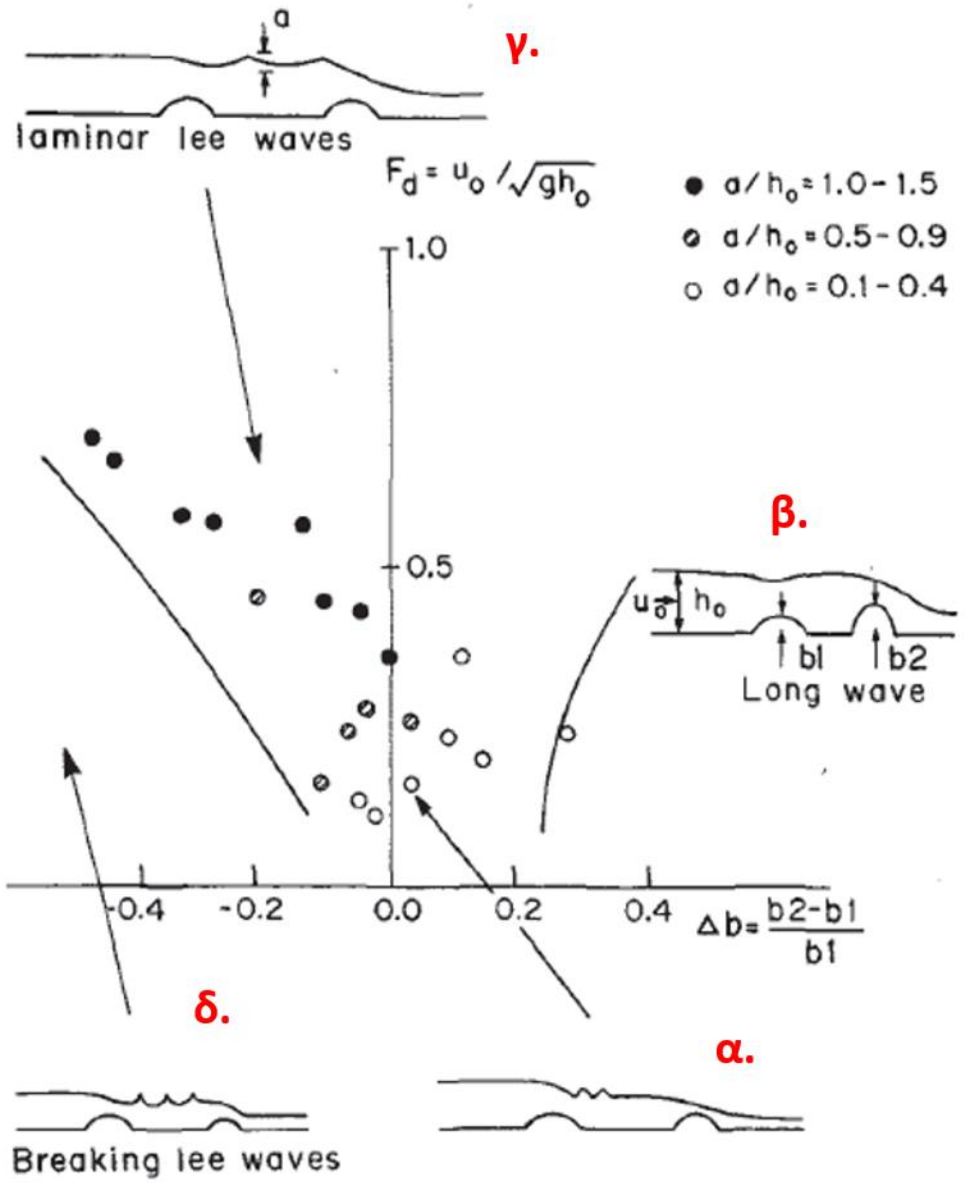


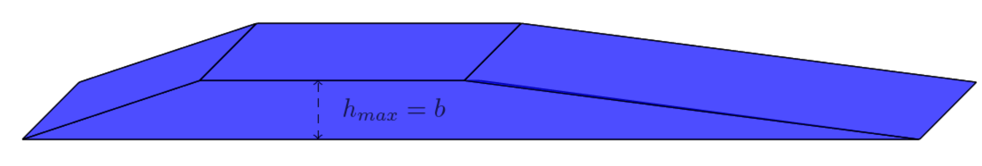
FIG. 5. Experimental results showing different flow regimes as function of $\Delta b = (b2 - b1)/b1$, and Froude number $Fd = u_0/(gh_0)^{1/2}$ measured upstream of first obstacle. The relative amplitude a/h_0 of the laminar wave nearest the upstream obstacle is indicated by the type of dot, as defined in the figure.

II) 1D free surface channel

Downstream gate

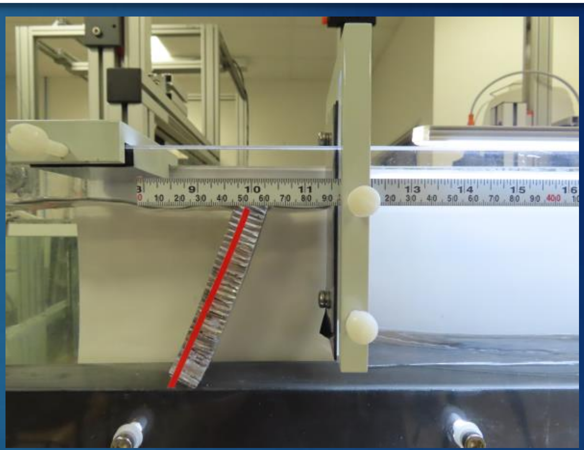
Flow inlet

pump



Type of asymmetric geometry used:
Length: 32.2 cm
Maximum heigth: 2.1 cm

Sluice gate

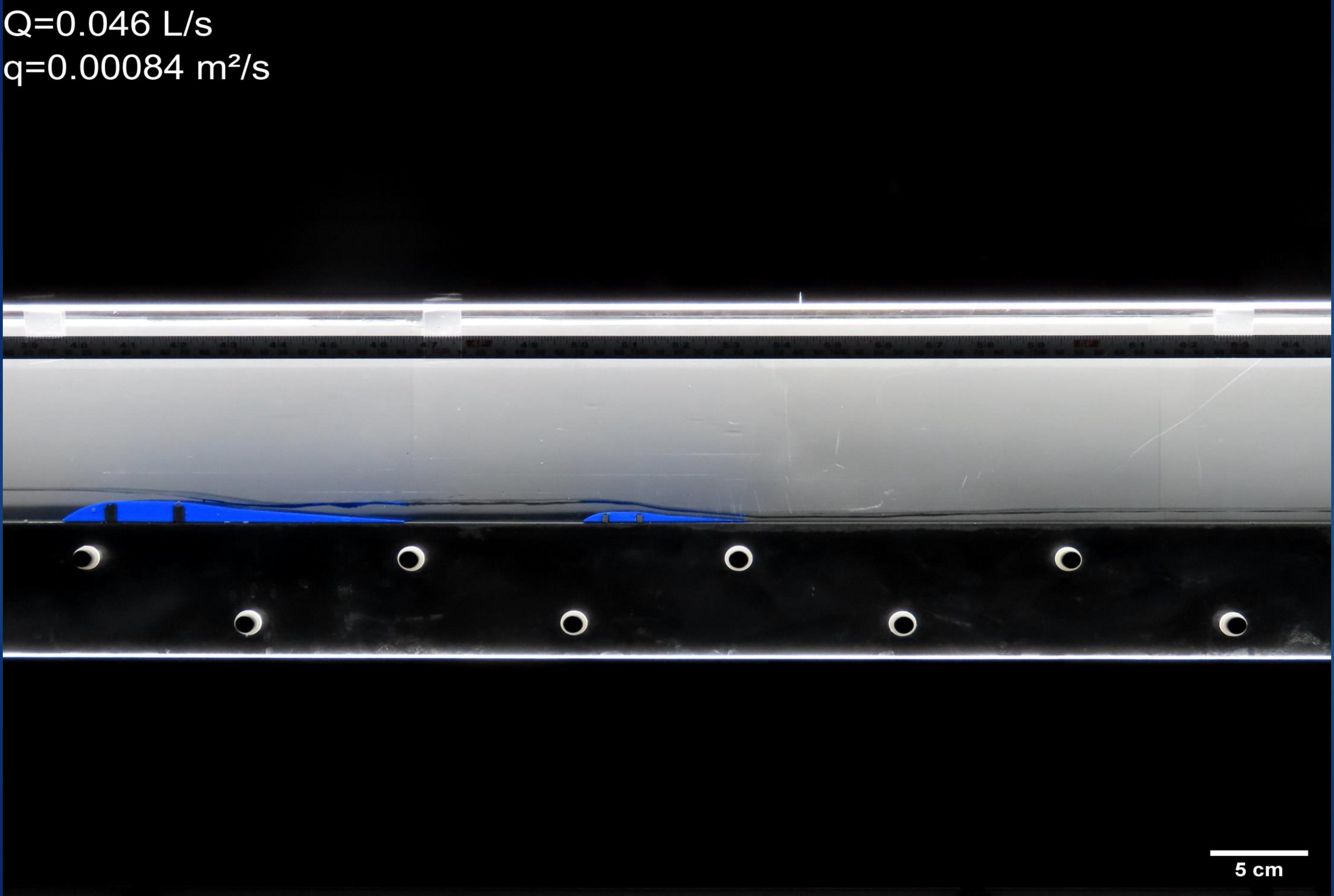


Hypotheses and experimental conditions:

- flow conservation
- $U_{upstream}$=Q/(Wh)
- No downstream condition (door open)
- No initial water level imposed
- Inter-obstacle distance set at 9.2 cm (arbitrary)
- Neglected boundary layer

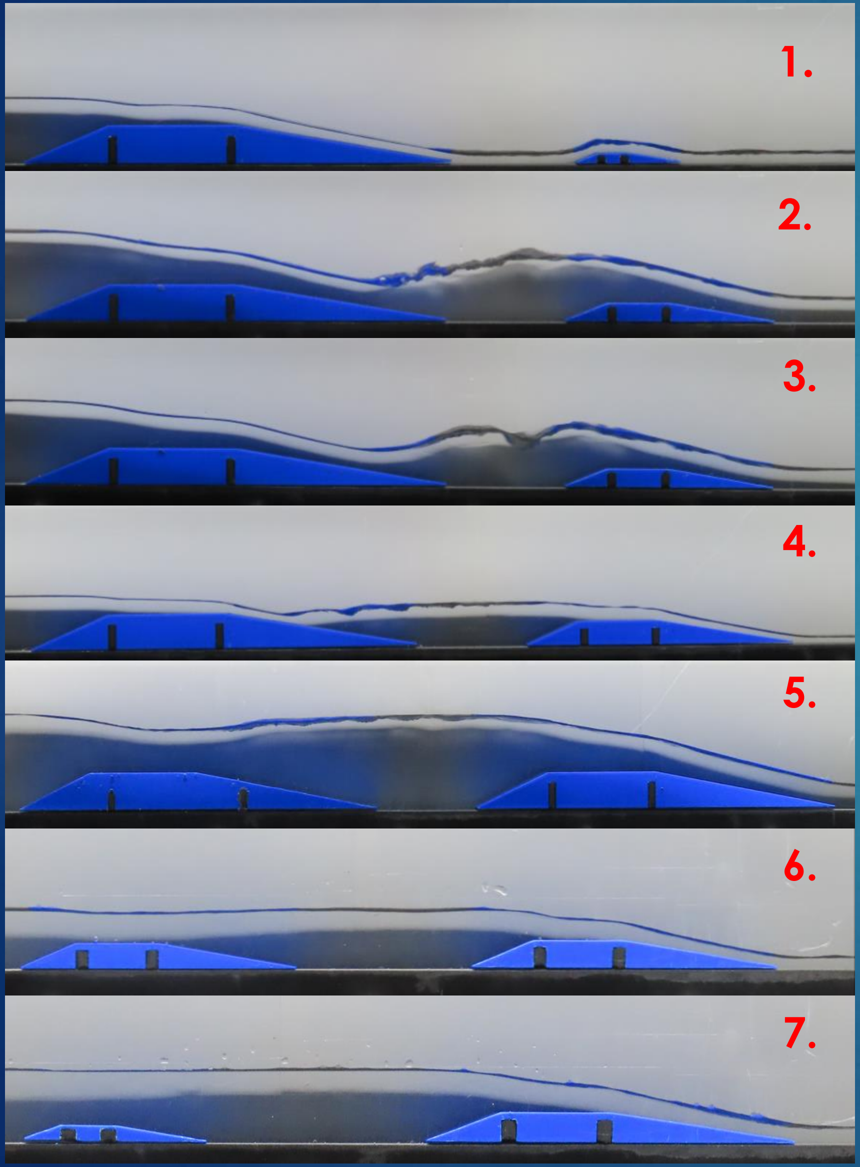
Channel characteristics:

- Length: L=2.5 m
- Wide: W=5.3 cm
- Range of the flow rate: 2 to 38 L/min
- Range of the flow rate: 0.0006 to 0.0115 m^2/s



The flow rate is increased from one image to the other.

III) Flow regimes

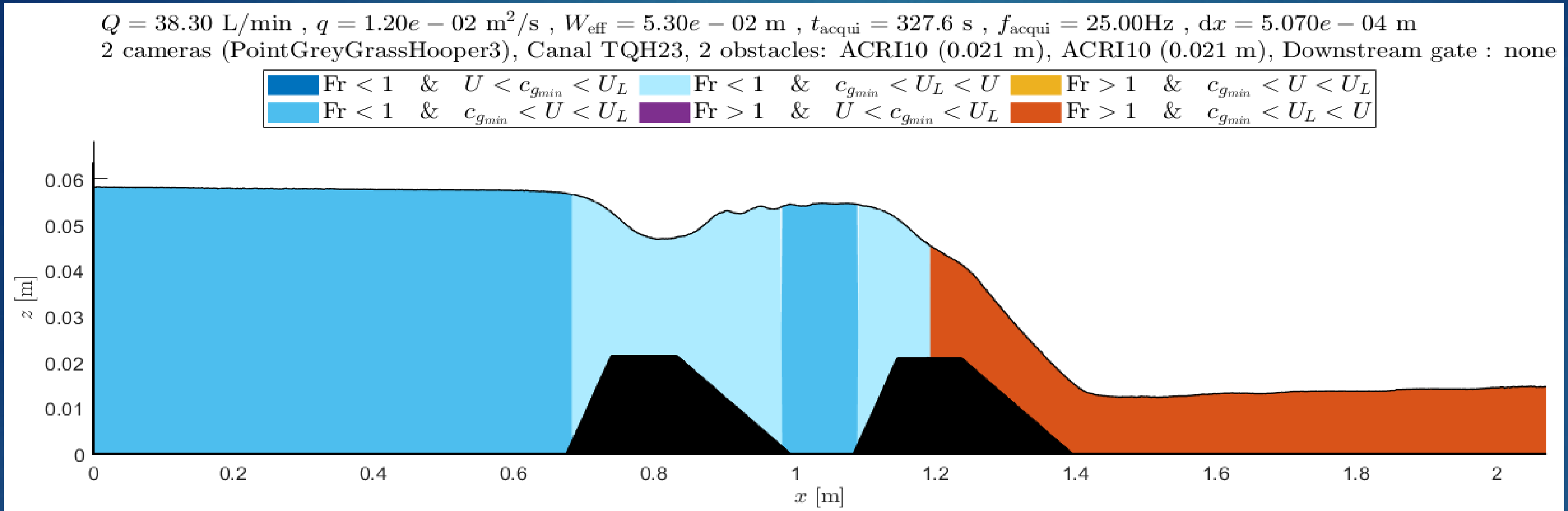


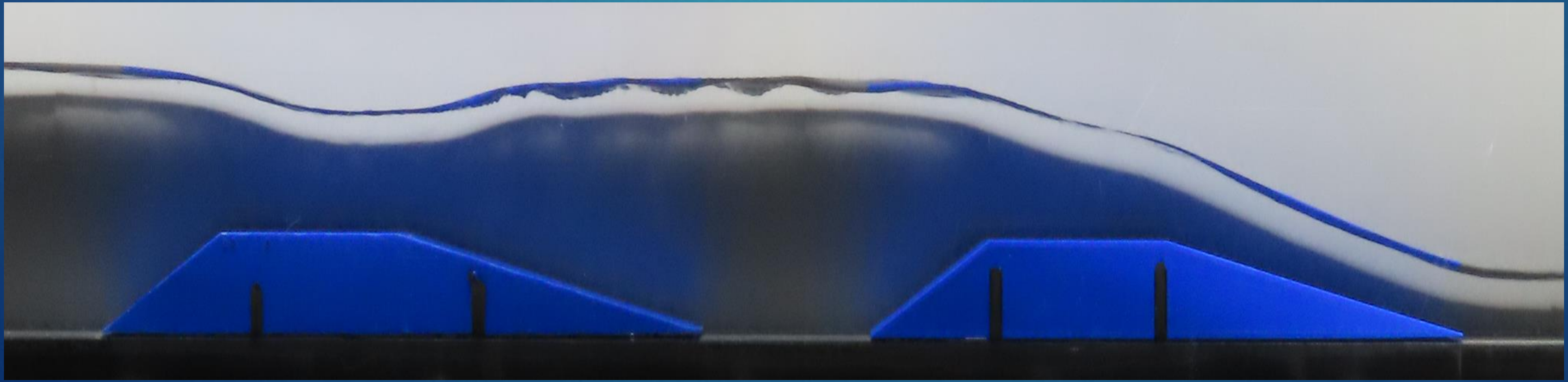
	Identification in the Pratt diagram	
1. T_a-S-S (Transcritical accelerating-Supercritical-Supercritical)	New regime	✗
2. T_a-B-T_a (Transcritical accelerating-Breaking- Transcritical accelerating)	New regime	✓
3. T_a-UB-T_a (Transcritical accelerating-Undular Breaking- Transcritical accelerating)	δ	✓
4. T_a-U*-T_a (Transcritical accelerating-Undular- Transcritical accelerating)	α	✓
5. D-U-T_a (Depression-Undular-Transcritical accelerating)	γ	✓
6. D-E-T_a (Depression-Emitting-Transcritical accelerating)	β	?
7. F-F-T_a (Flat-Flat-Transcritical accelerating)	New regime	✗

According to Coutant and Parentani (2014)[13] and Euvé (2016)[5] (regimes 2–5)

Germain Rousseaux et Hamid Kellay (2020), Introduction of the nomenclature with one obstacle : F, D, U, B, etc[14]

Example of interface extraction on the regime 5.b) (D-U-T_a) :





$$U(x) = \frac{Q}{W h(x)}$$

$$c_{gmin} = \min_{k \in \mathbb{R}_+} (v_g)$$

$$c_{gmin} \underset{kh>>1}{=} \frac{\sqrt{3}}{\sqrt[4]{2\sqrt{3}+3}} \sqrt[4]{\frac{\gamma g}{\rho}}$$

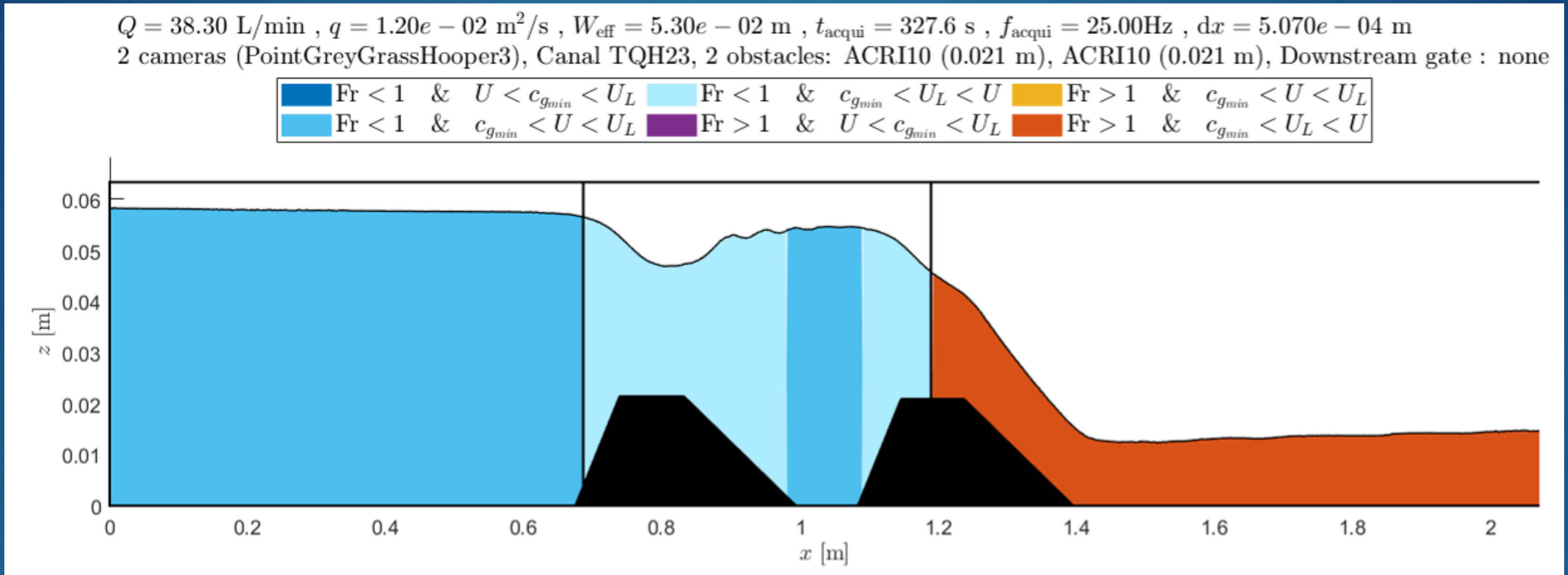
$$U_L = \min_{k \in \mathbb{R}_+} (v_\varphi)$$

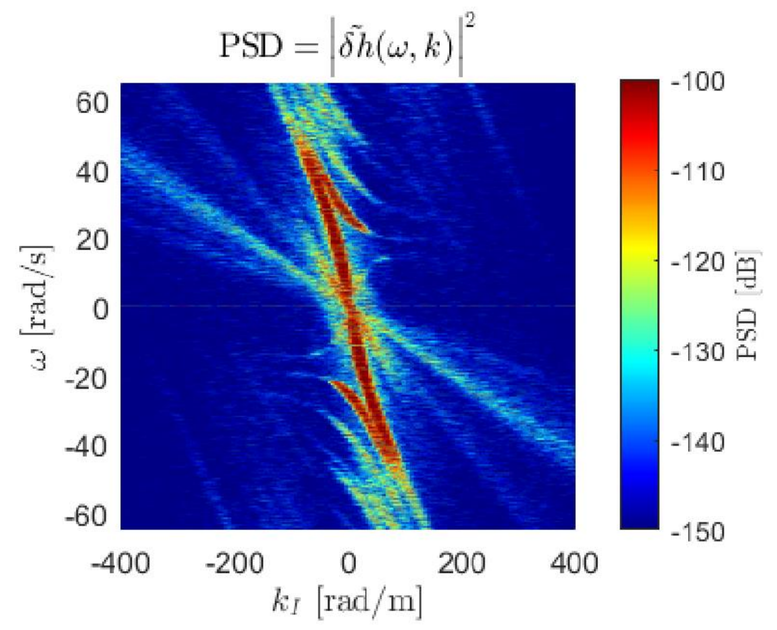
$$U_L \underset{kh>>1}{=} \sqrt{2} \sqrt[4]{\frac{\gamma g}{\rho}}$$

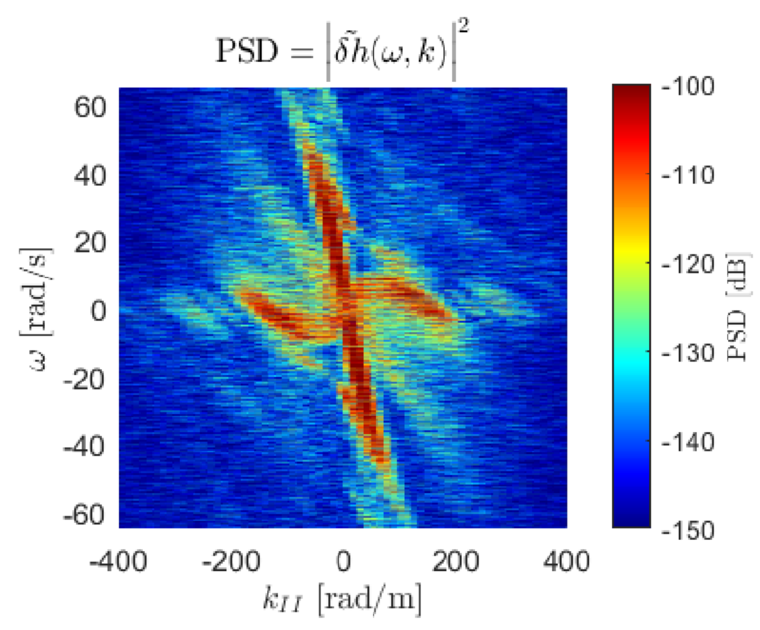
= possible existence of negative modes

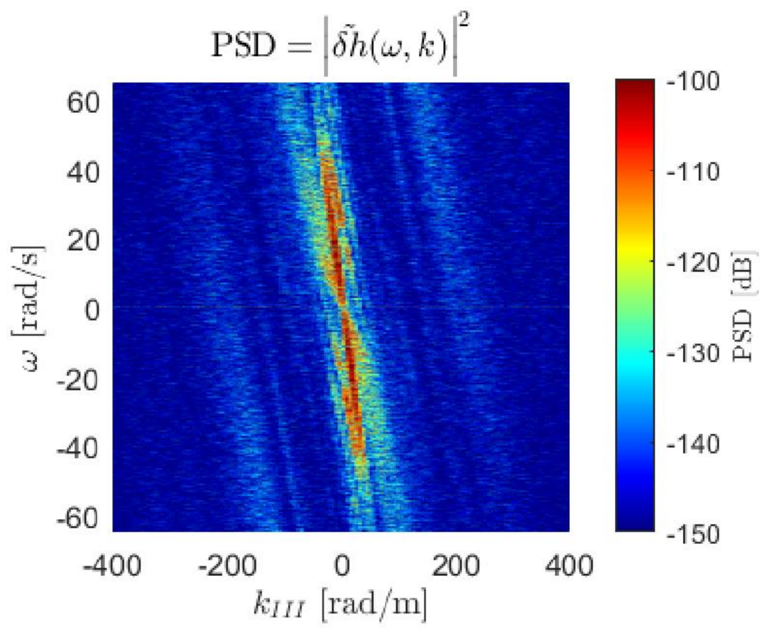
$$\left(\omega - \overrightarrow{U}\cdot\overrightarrow{k}\right)^2 = gk\left(1+\frac{\gamma}{\rho g}k^2\right)\operatorname{th}(kh)$$

Can modes be measured?

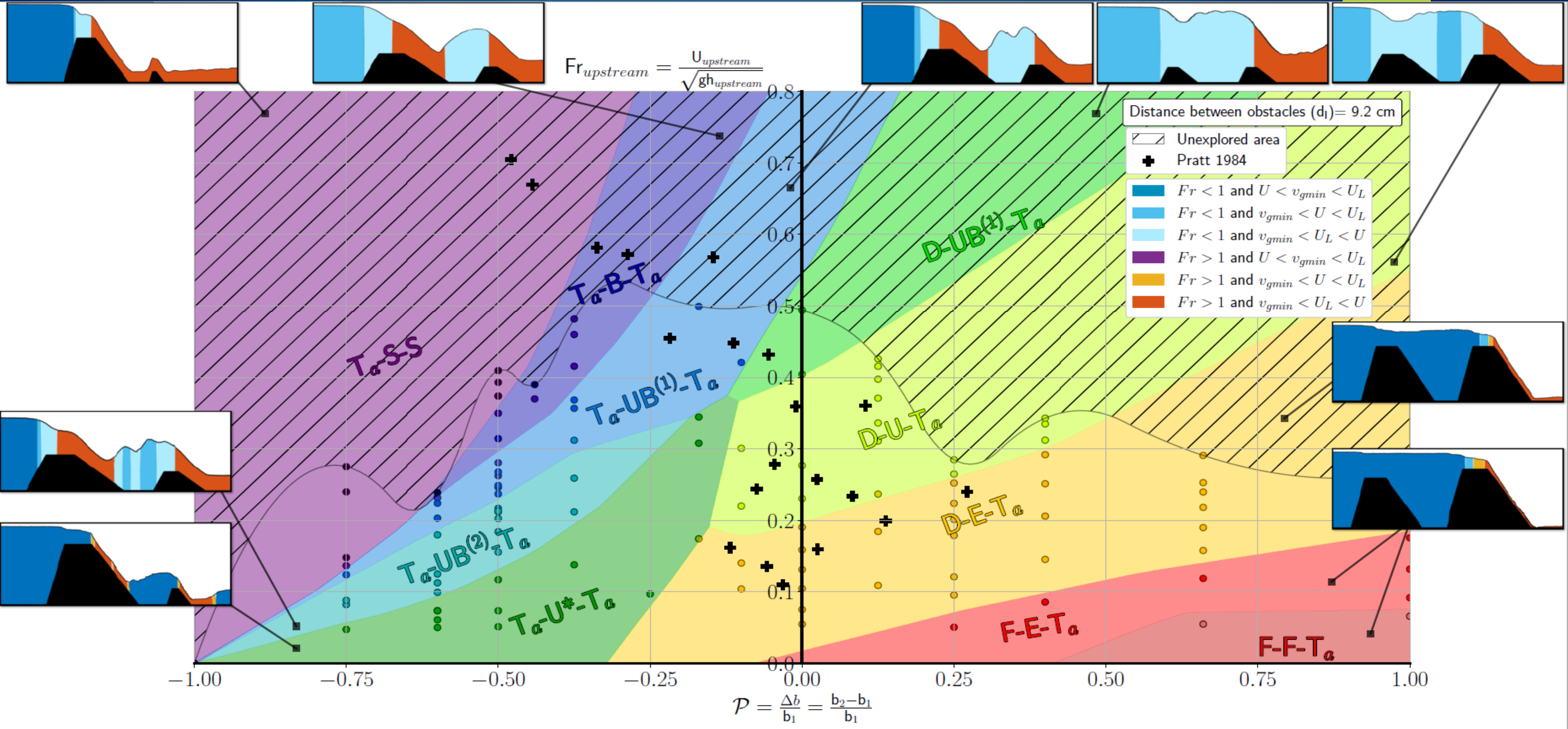








Improved Pratt diagram ($Fr_{upstream}(\mathcal{P})$)

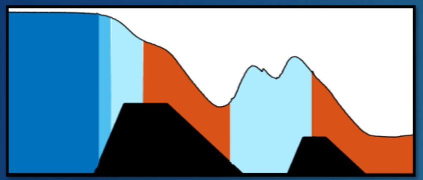
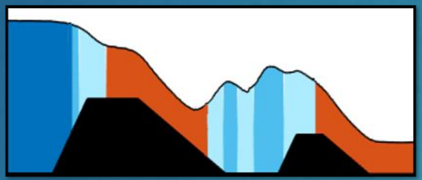
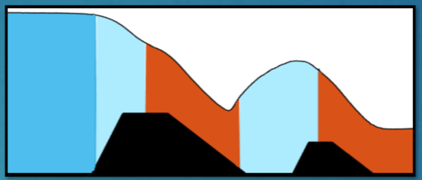
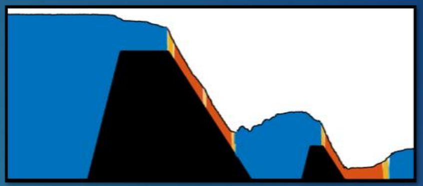
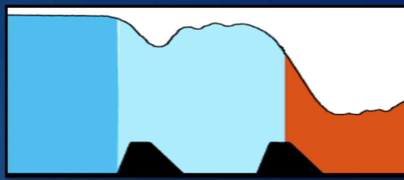


Inter-obstacles distance: 9.2 cm
without downstream condition

Number of experimental points to train the neural network: 119

Summary

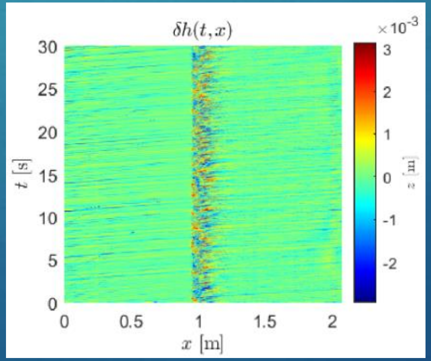
1. You need two stable horizons
2. A superluminal or subluminal dispersive correction
3. A trapping cavity with a flow regime compatible with the dispersive regime
4. Mixing of positive and negative modes

	T_a-$UB^{(1)}$-T_a	T_a-$UB^{(2)}$-T_a	T_a-B-T_a	T_a-U^*-T_a	D-$UB^{(1)}$-T_a
					
1	✓	✓	✗	✓	✓
2	✓	✓	✓	✓	✓
3	✓	✓	✓	✓	✓
4	?	?	?	?	?

T_a-$UB^{(1)}$-T_a: Closest scenario to general relativity Because it is possible to have negative energy waves in the cyan subliminal cavity.

T_a-$UB^{(2)}$-T_a: In this scenario, the hawking radiation can be blocked because of capillary dispersion at small scales

T_a-B-T_a: Presence of two horizons but the white horizon is unsteady



T_a-U^*-T_a: two superluminal red cavities on both obstacles are formed but damping will kill the superluminal capillary waves.

D-$UB^{(1)}$-T_a: The undulation is trapped between a non-dispersive downstream horizon and a dispersive Upstream horizon.

Work in progress and perspectives:

1) Reproduction of the Nice experiments (Rousseaux et al (2008)[15]), in the linear regime:
Objective: To show that Hawking radiation can be confined to a zone, because of the dispersive response of the system: the radiation would not spread out to infinity;

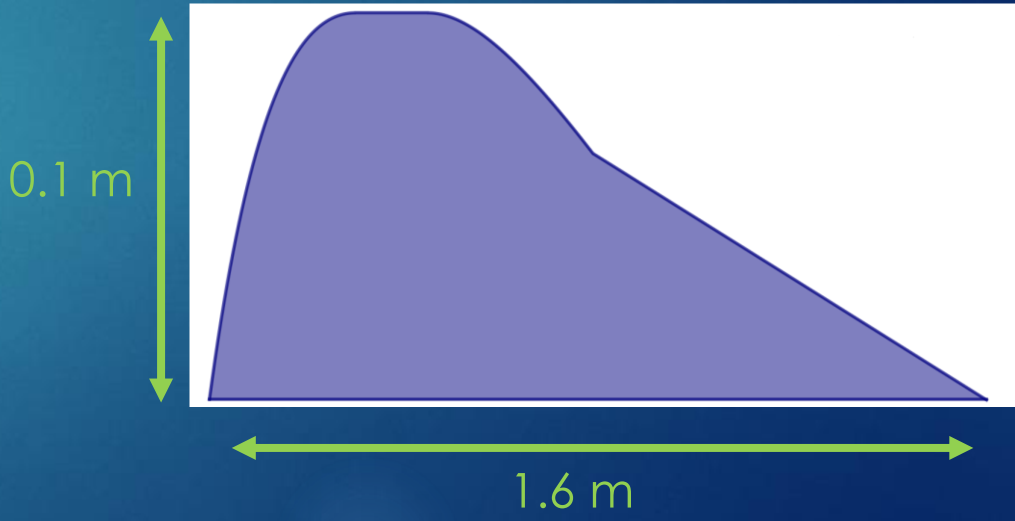
2) Corollary: radiation confinement could lead to a LASER effect with dispersive horizons.

The experimental setup

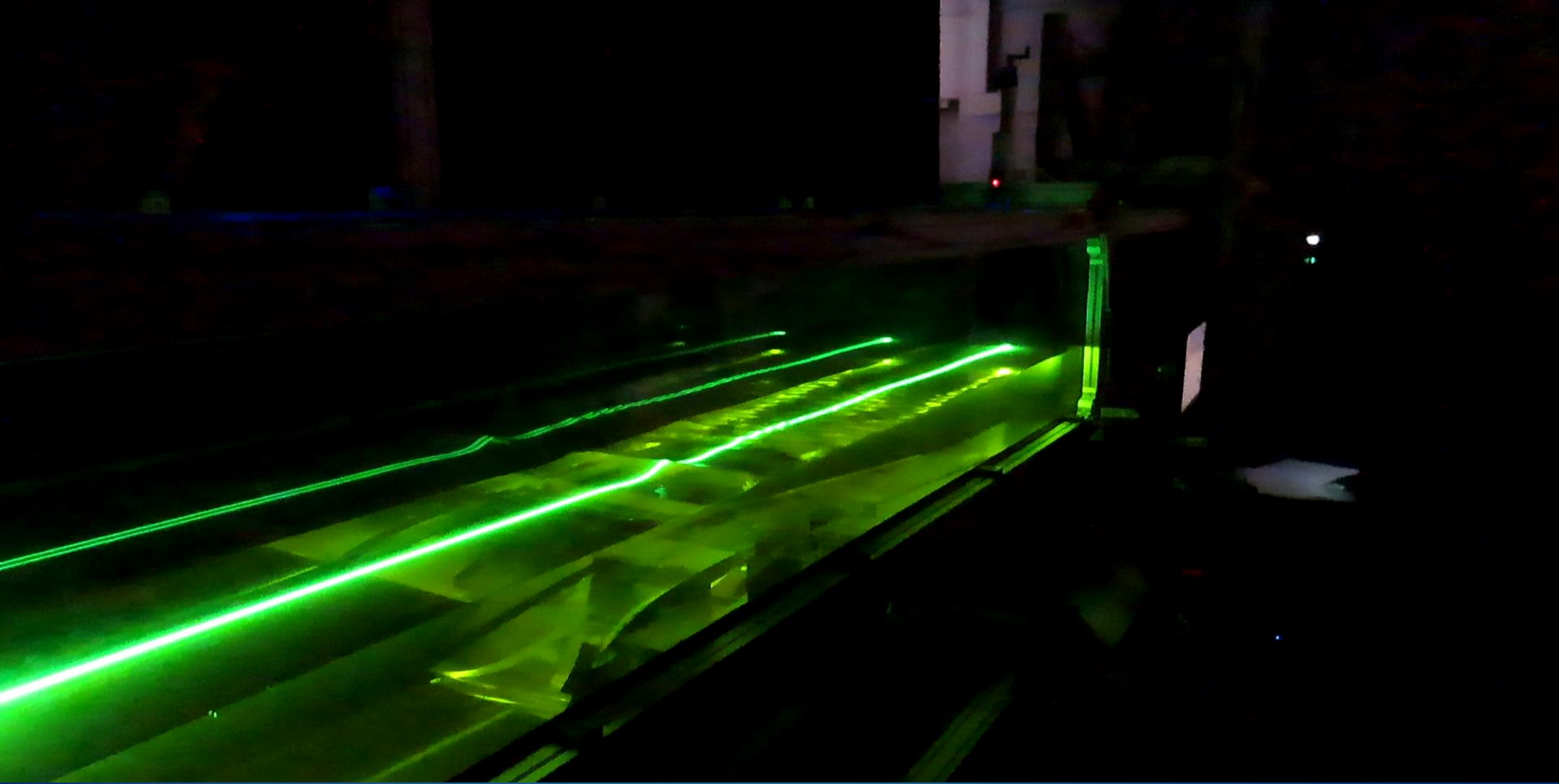
Euvé et al (2016) [16]

Characteristics:

- Wide: W=38.5 cm
- Length: L=7 m
- Range of the flow rate: 0.004 to 0.156 m^2/s
- Laser power: 300mW
- Laser wavelength: 473 nm



Weinfurtner et al (2011) [17]



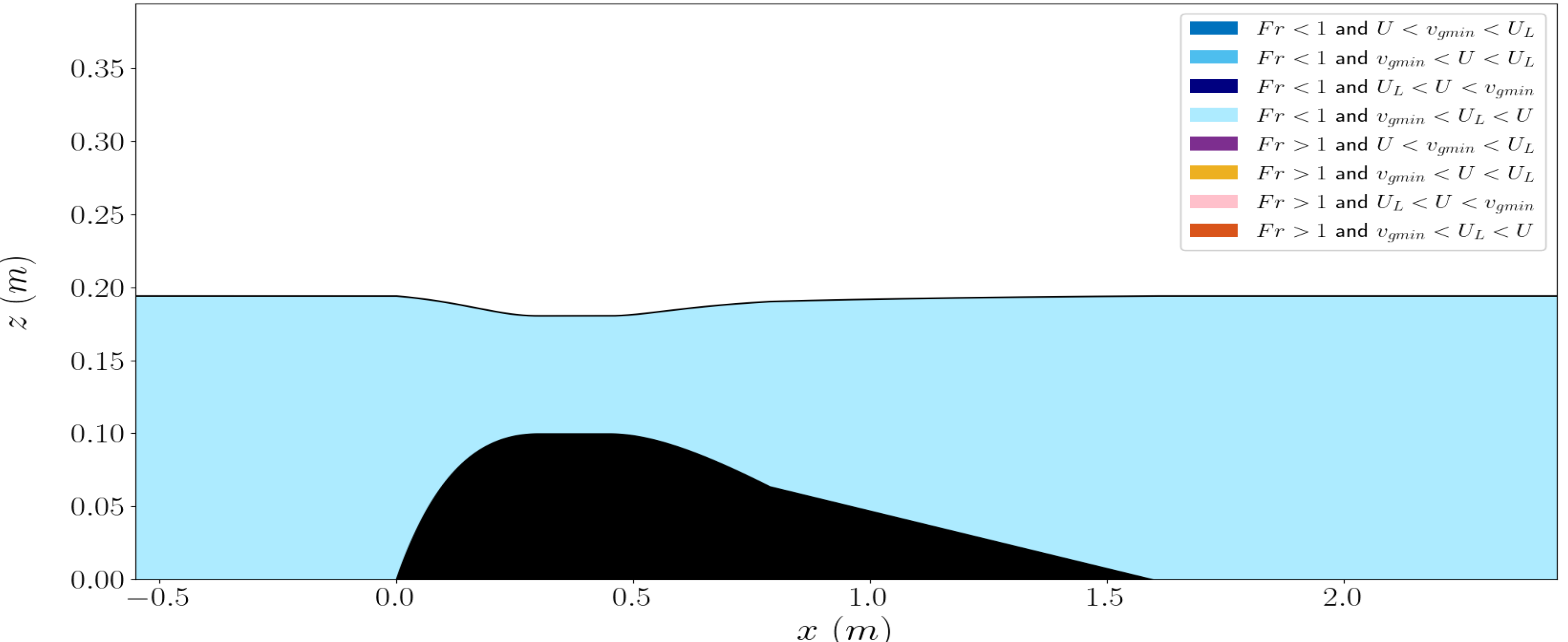
<u>Example</u>: $Q = 745.2\ L.min^{-1}, q = 0.032\ m^2.s^{-1}, h_{\text{up}} = 20.5\ cm, a_{\text{wave maker}} = 10\ mm$

Simulation of Vancouver-type regime: Weinfurtner et al (2011) [17]

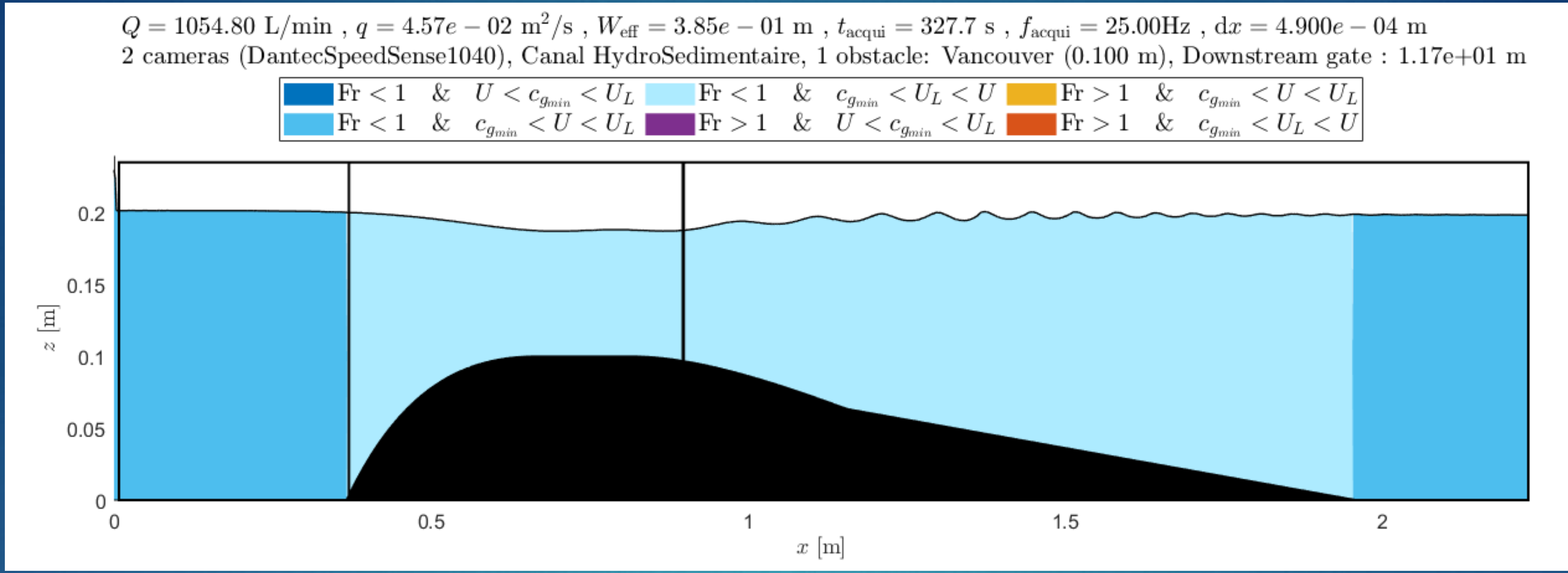
Height model based on stationary Saint-Venant equations:

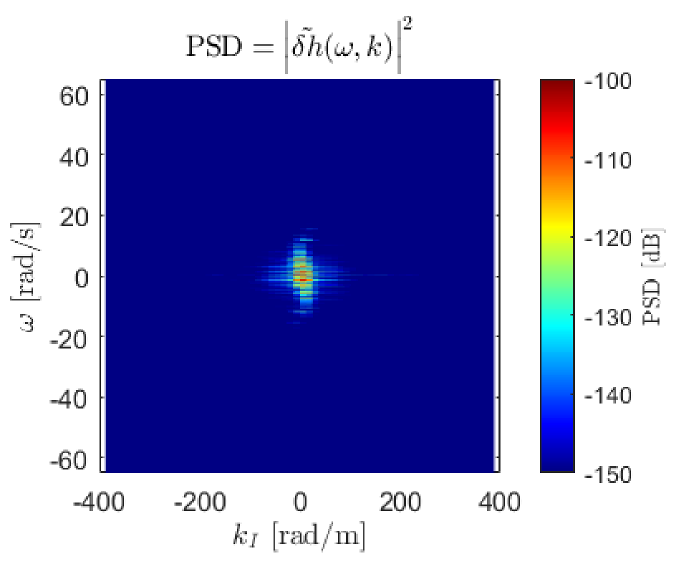
$$h(x) = \frac{h_{\text{up}}}{3}\left(1 + \frac{Fr_{\text{up}}^2}{2} - \frac{b(x)}{h_{\text{up}}}\right) + \frac{2h_{\text{up}}}{3}\left(1 + \frac{Fr_{\text{up}}^2}{2} - \frac{b(x)}{h_{\text{up}}}\right)\cos\left(\frac{1}{3}\,\text{Arccos}\left(1 - \frac{\frac{27}{4}Fr_{\text{up}}^2}{\left(1 + \frac{Fr_{\text{up}}^2}{2} - \frac{b(x)}{h_{\text{up}}}\right)^3}\right)\right)$$

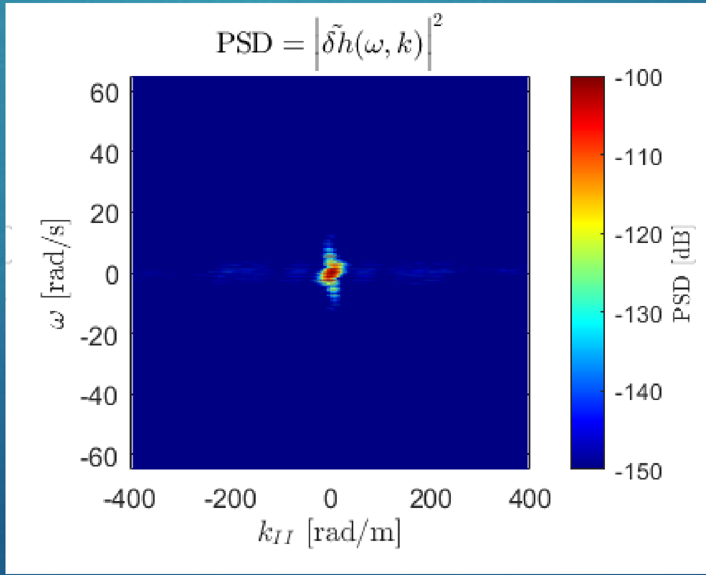
Example: $h_{\text{up}} = 0.194\ m, q = 0.046\ m^2.s^{-1}$

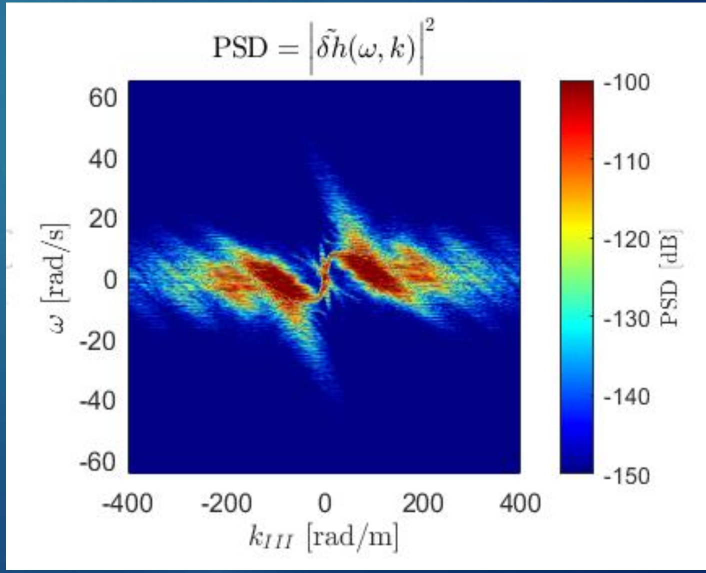


Vancouver-type regime, preliminary results:







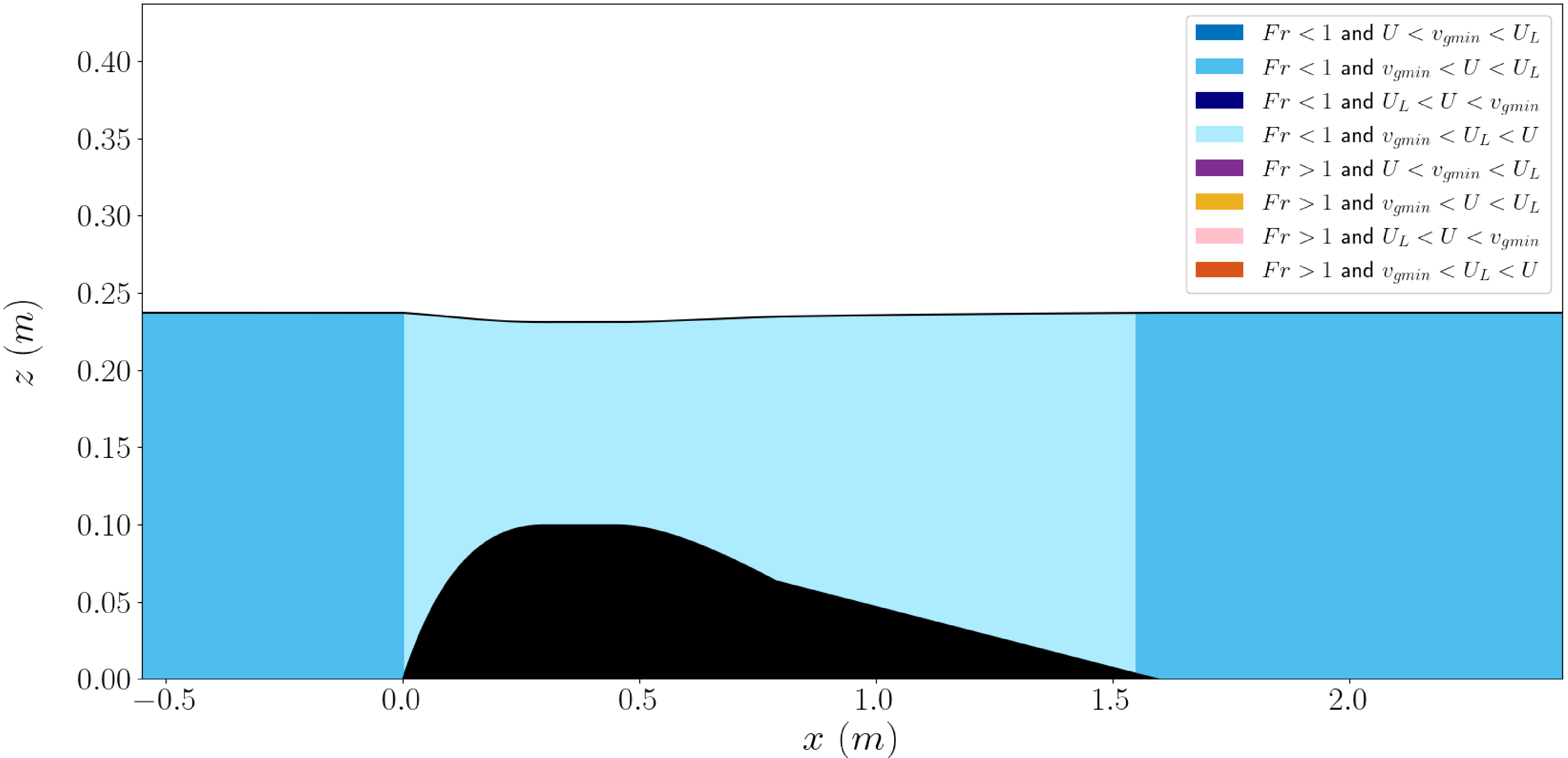


Simulation of Nice-type regime:

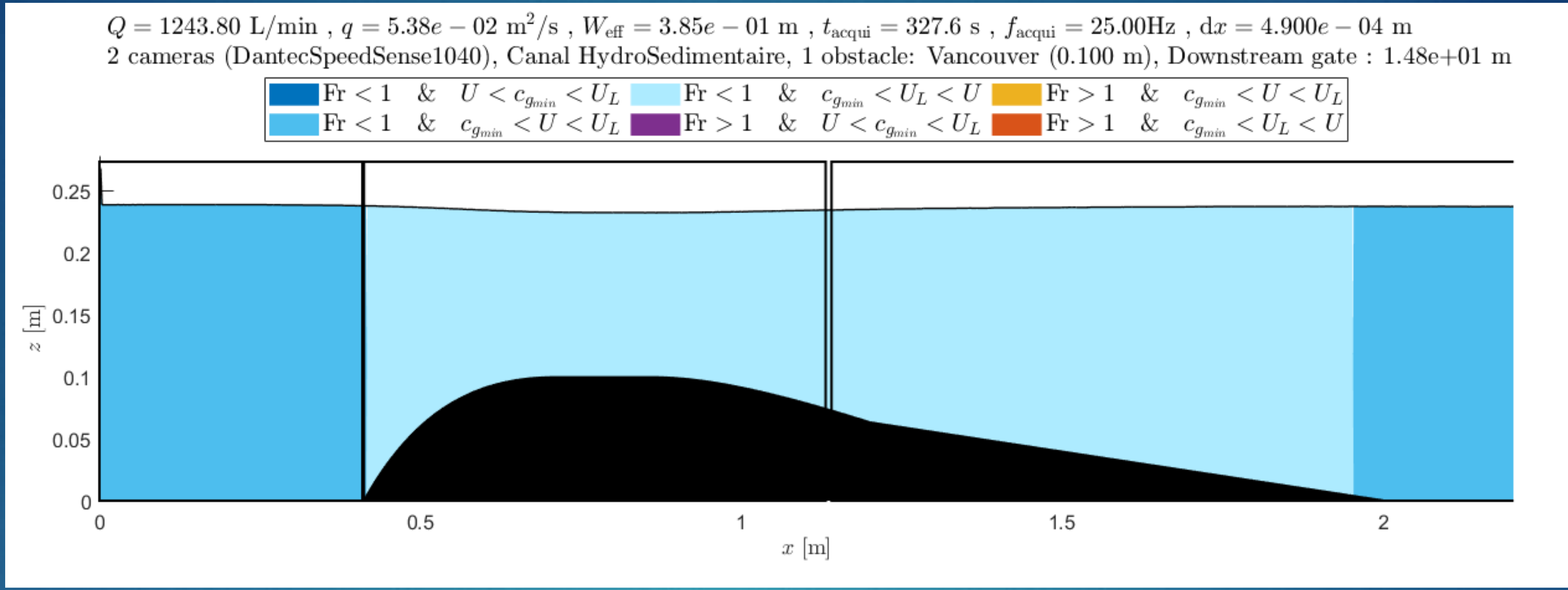
Rousseaux et al (2008) [15]

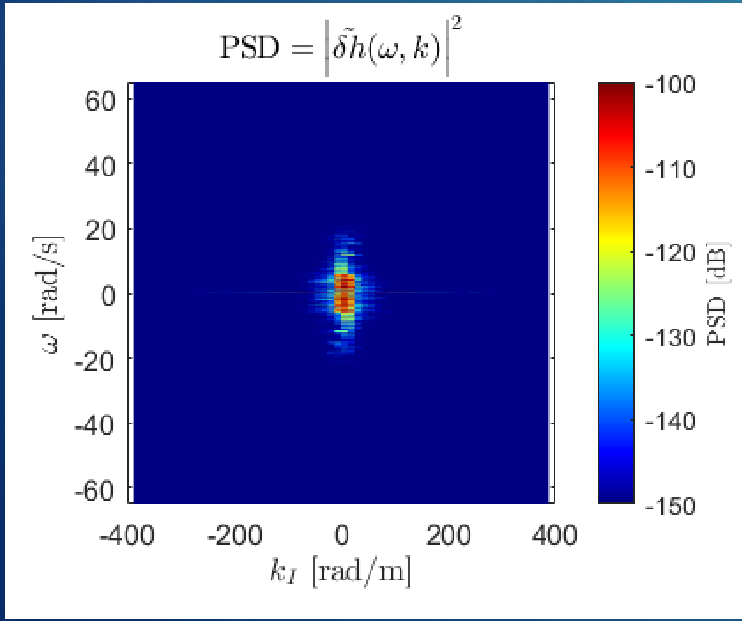
Definition: Nice-type regime=Pre-Vancouver type regime without ondulations

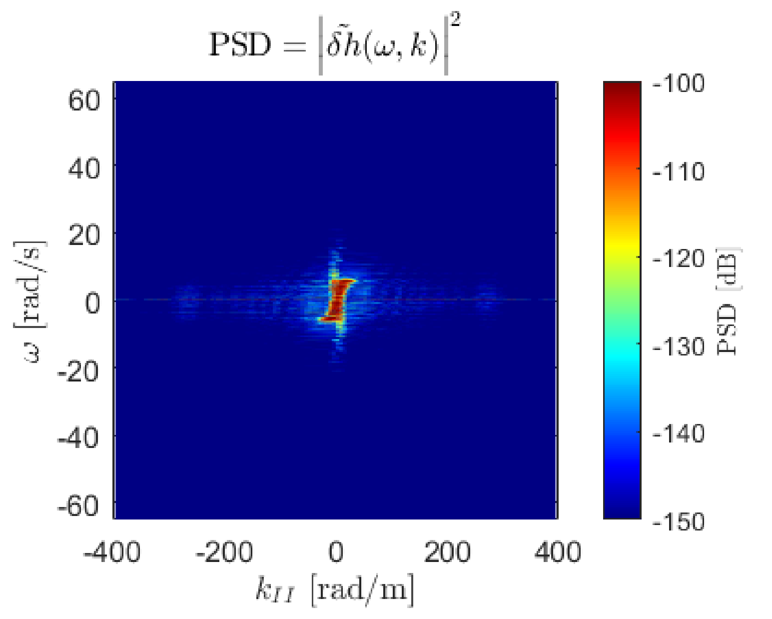
Example: $h_{\text{up}} = 0.237\, m, q = 0.054\, m^2.s^{-1}$

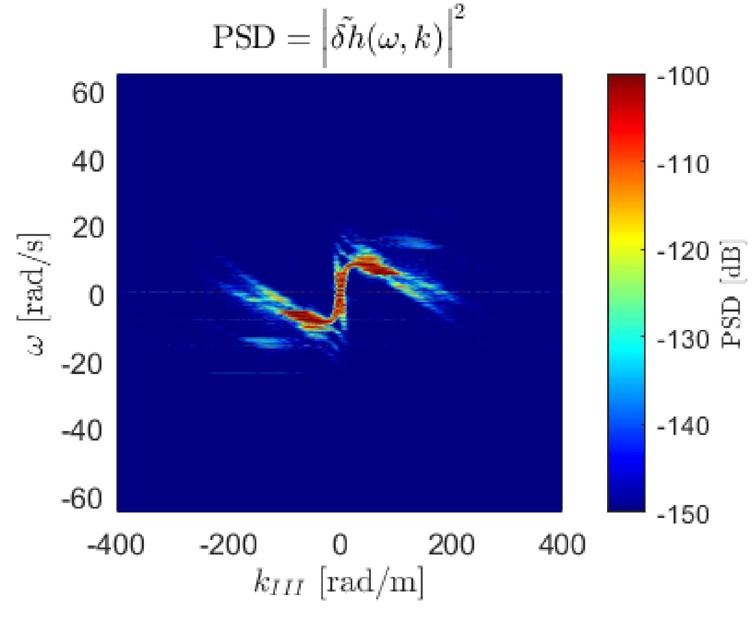


Nice-type regime, preliminary results :





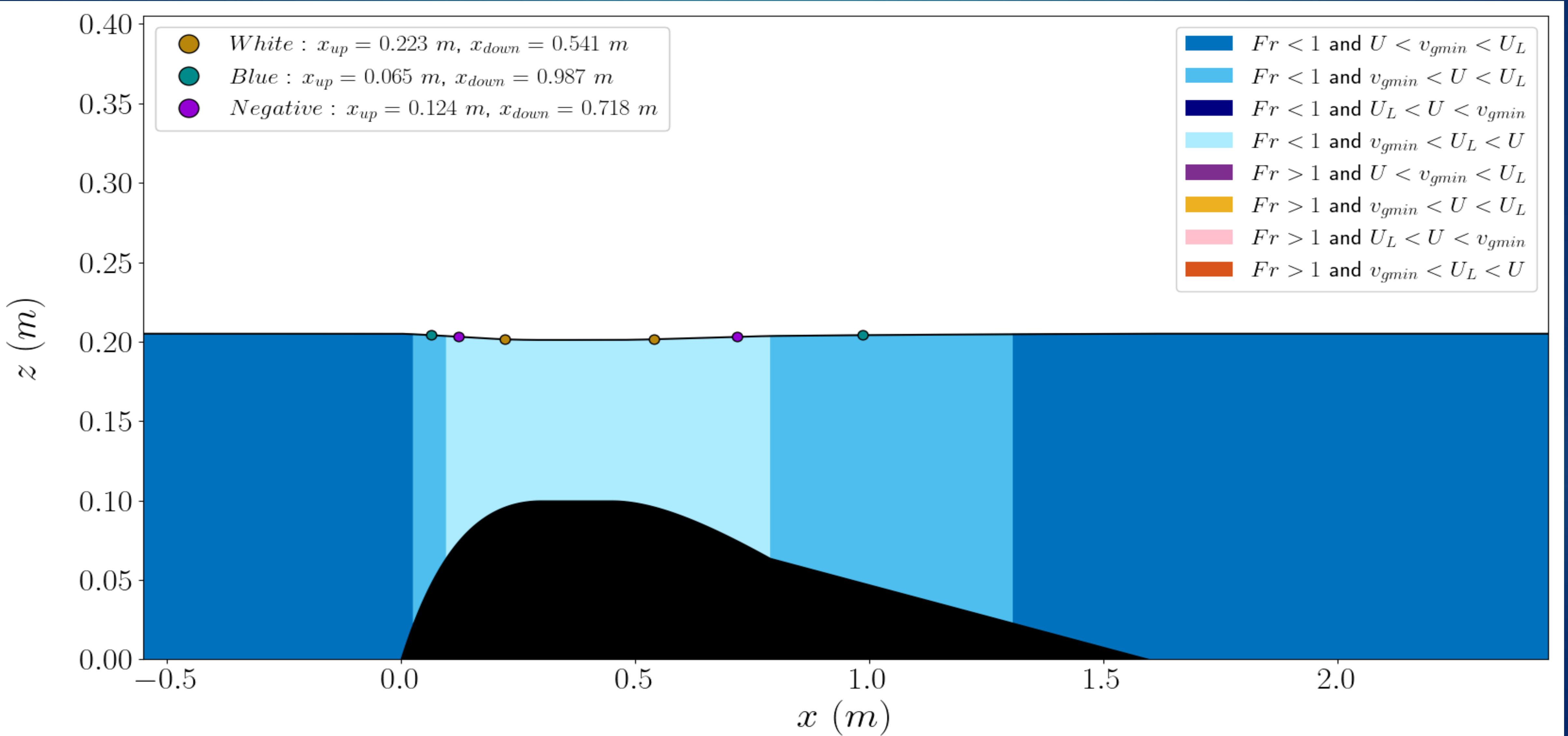




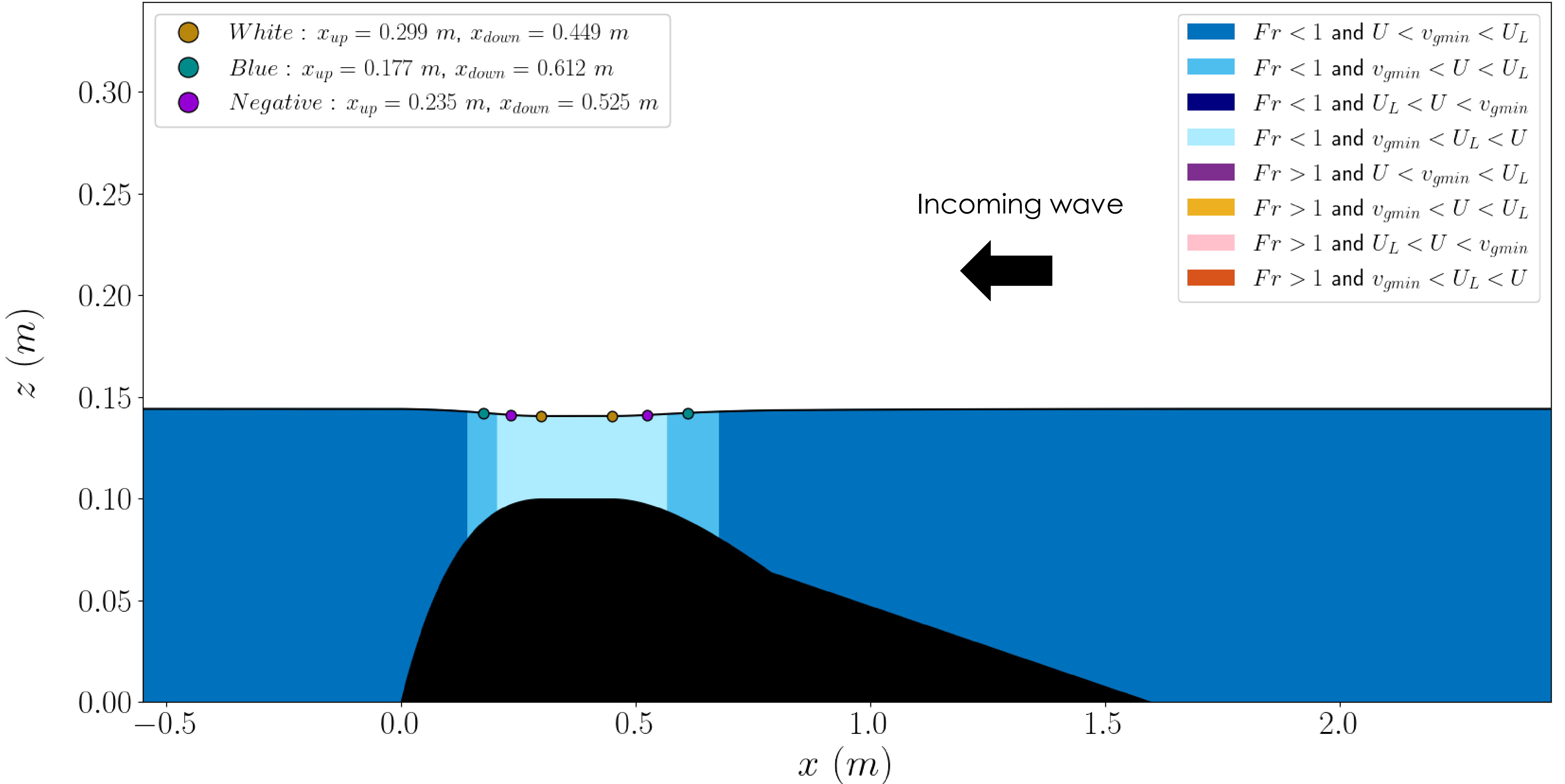
Dispersive horizons = Blocking point when $Fr_d = \frac{U}{v_g} = 1$

Germain Rousseaux et al (2010) [15]

Example: $T = 0.784\, s, h_{\text{up}} = 0.205\, m, q = 0.032\, m^2.s^{-1}$



Laser effect between negative blocking horizons ?



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[1] William G. Unruh. Experimental black-hole evaporation? Physical Review Letters, 46(21):1351,1981.

[2] Stephen W. Hawking. Black hole explosions? Nature,248(5443):30-31,1974.

[3] Matt Visser(1998). Acoustic black holes: horizons, ergospheres and Hawking radiation. *Classical and Quantum Gravity*, *15*(6), 1767.

[4] Weinfurtner, Silke, et al. "Measurement of stimulated Hawking emission in an analogue system." *Physical review letters* 106.2 (2011): 021302.

[5]Léo-Paul Euvé, Florent Michel, Renaud Parentani, Thomas G. Philbin, and Germain Rousseaux. Observationof noise correlated by the hawking effect in a water tank. Physical review letters, 117(12) :121301, 2016.

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[14] Germain Rousseaux & Hamid Kellay, Classical hydrodynamics for analogue space–times: open channel flows and thin films. Philosophical Transactions of the Royal Society A, Volume 378, Issue 2177, 20190233, July 2020.

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[15] Germain Rousseaux, Christian Mathis, Philippe Maïssa, Thomas G. Philbin and Ulf Leonhardt. Observation of negative-frequency waves in a water tank: a classical analogue to the Hawking effect?. *New Journal of Physics*, *10*(5), 053015, 2008

[16] Léo-Paul Euvé, Florent Michel, Renaud Parentani , Thomas G, Philbin and Germain Rousseaux. Observation of noise correlated by the Hawking effect in a water tank. *Physical review letters*, *117*(12), 121301, 2016

[17] Silke Weinfurtner, Edmund W. Tedford, Matthew C. Penrice, Wiliam G. Unruh, & Gregory A. Lawrence. Measurement of stimulated Hawking emission in an analogue system. *Physical review letters*, *106*(2), 021302, 2011

How to construct the improved Pratt diagram?

```
from sklearn.neural_network import MLPClassifier
```

1. I define labels for the dataset
2. I train the network with the data
3. After training, the network predicts a label for the whole plan

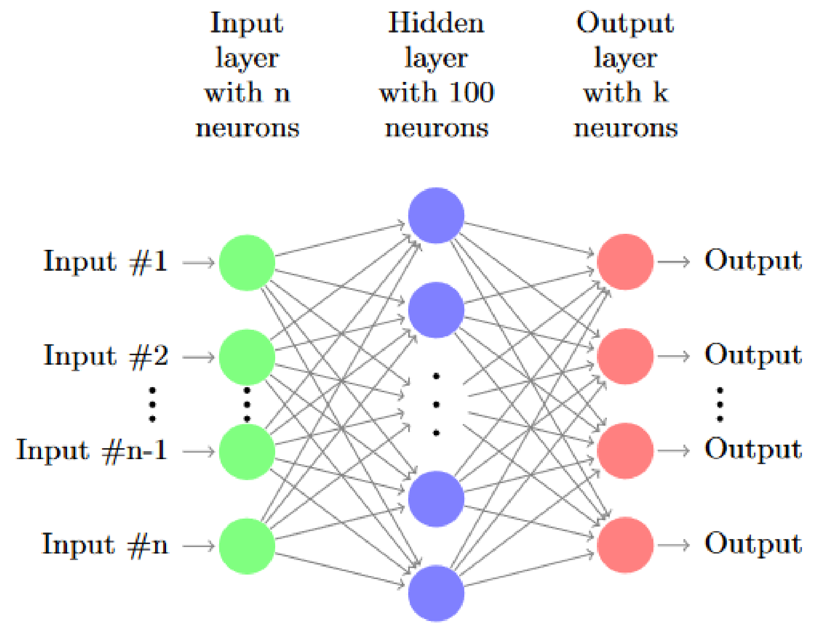


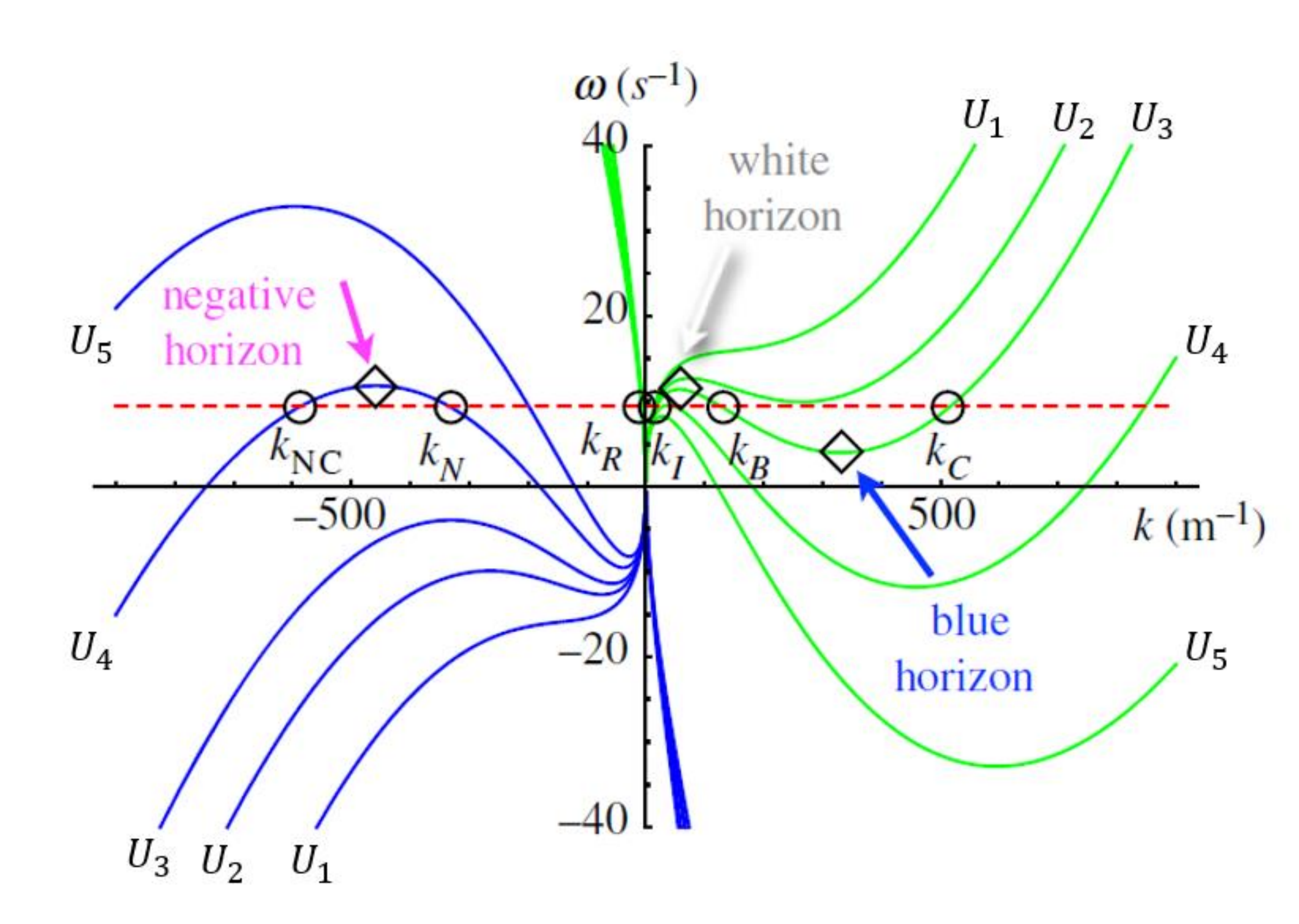
Figure 11. Architecture of the neural network with n the number of points in the training set and k the number of groups (here the number of hydrodynamic regimes).

<u>Parameters used :</u>

- 3 layers, 1 hidden layer
- 100 neurons in the hidden layer
- Maximum iteration : 3000

- Default setting:
  1. Cost function: entropy function
  2. Solver: adam
  3. Activation function: the rectified linear unit function

$$\left(\omega - \overrightarrow{U} \cdot \overrightarrow{k}\right)^2 = gk\left(1 + \frac{\gamma}{\rho g}k^2\right) \operatorname{th}(kh)$$



Analogue Wormholes and Black Hole LASER Effect in Hydrodynamics

Cédric Peloquin,[1] Léo-Paul Euvé,[2] Thomas Philbin,[3] and Germain Rousseaux[2]

[1] *Université François Rabelais de Tours, 60 Rue du Plat d'Etain, 37000 Tours, France*
[2] *Pprime Institute, UPR 3346, CNRS - Université de Poitiers - ISAE ENSMA,*
11 Boulevard Marie et Pierre Curie, Téléport 2, BP 30179, 86962 Futuroscope Cedex, France
[3] *Physics and Astronomy Department, University of Exeter, Stocker Road, Exeter EX4 4QL, United Kingdom*

Viscous dissipation of surface waves and its relevance to analogue gravity experiments

S. Robertson[1*], G. Rousseaux[2]

1 Laboratoire de Physique Théorique, UMR 8627, CNRS,
Univ. Paris-Sud, Université Paris-Saclay, 91405 Orsay, France
2 Institut Pprime, UPR 3346, CNRS-Université de Poitiers-ISAE ENSMA,
11 Boulevard Marie et Pierre Curie-Téléport 2, BP 30179, 86962 Futuroscope, France
* scott.robertson@th.u-psud.fr

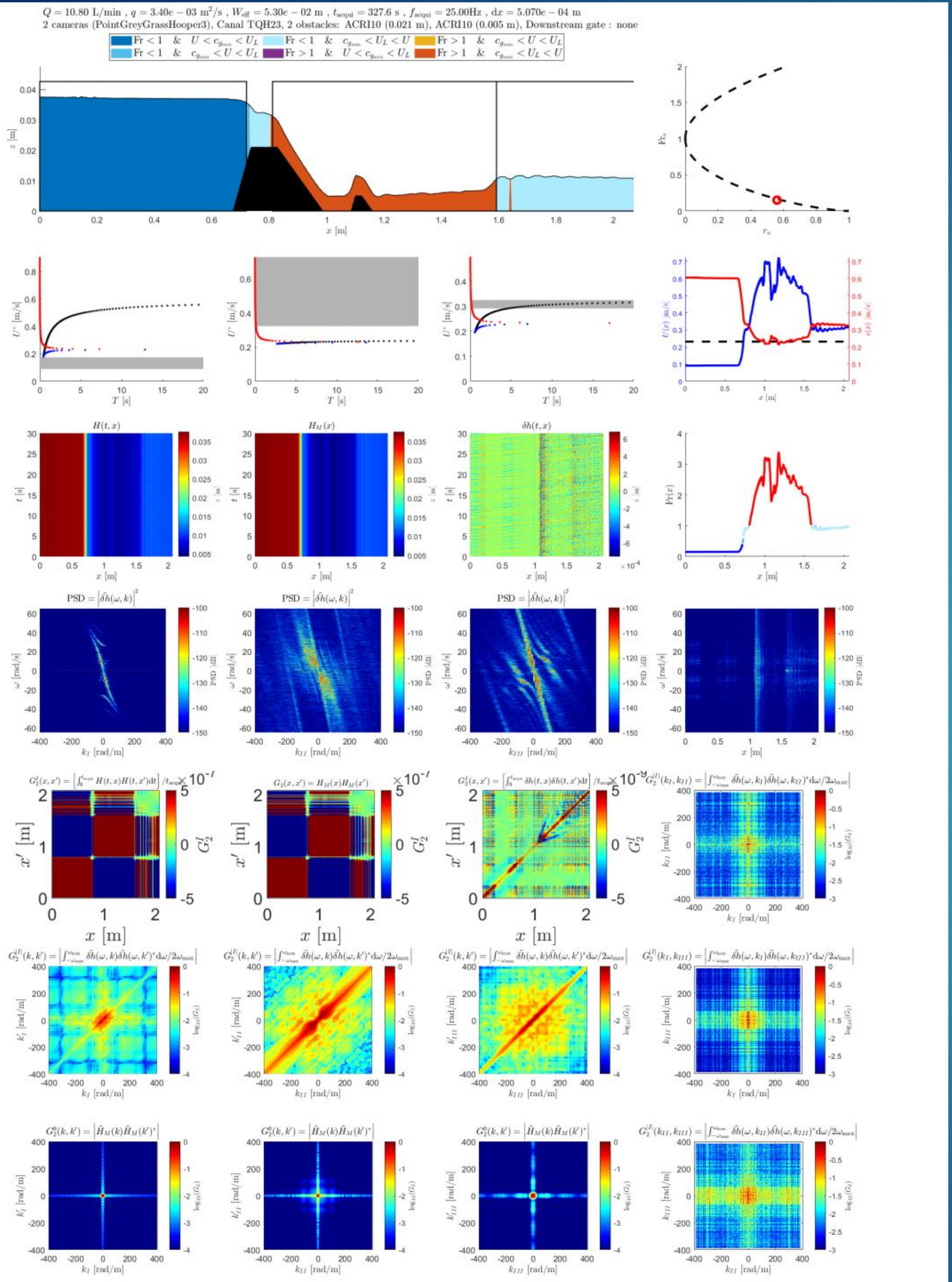
More difficult to detect the black hole laser effect!

Confirmation of stimulated Hawking radiation, but not of black hole lasing

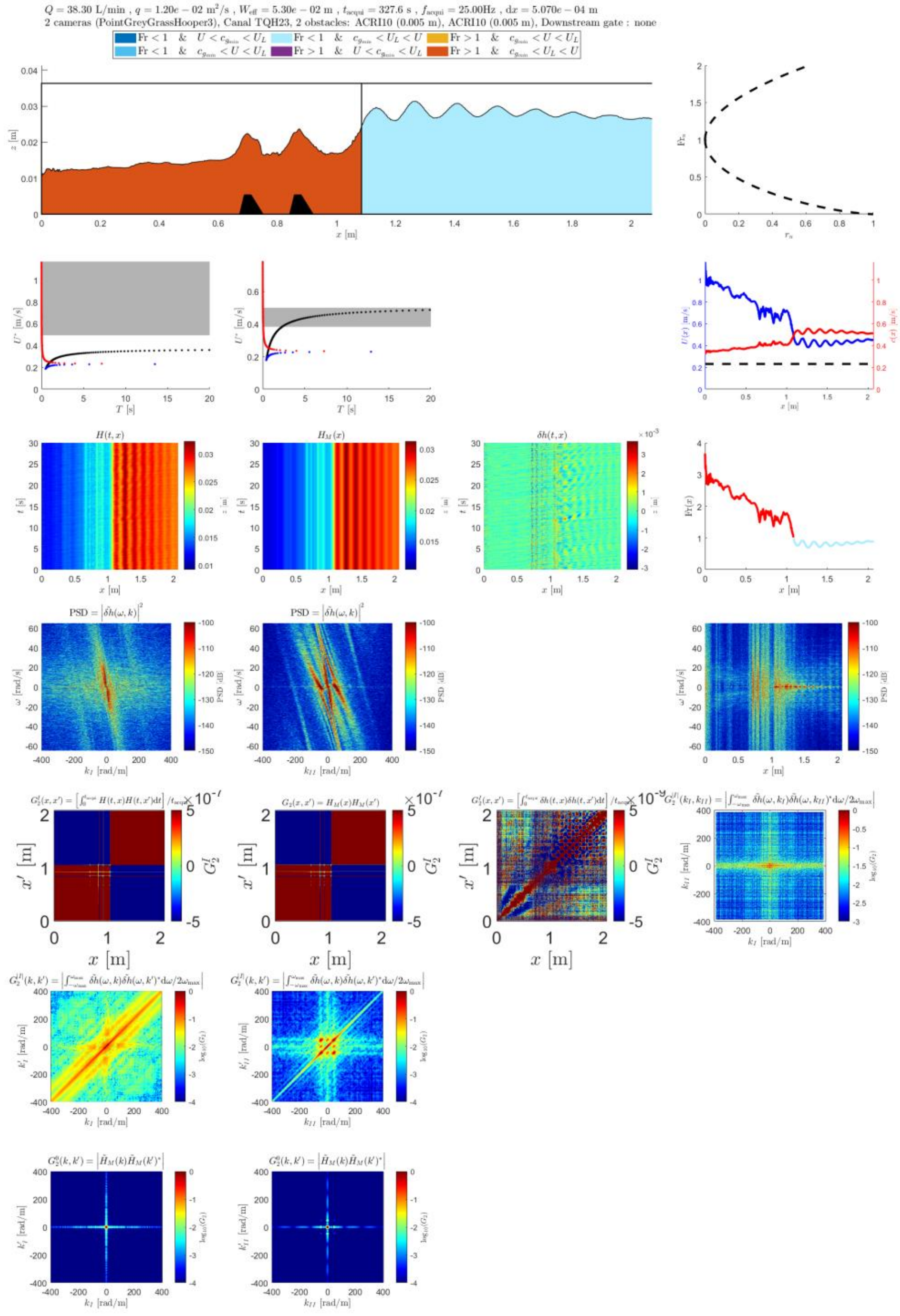
Jeff Steinhauer[1]

[1] *Department of Physics, Technion—Israel Institute of Technology, Technion City, Haifa 32000, Israel*

(Dated: October 14, 2021)



Q = 10.80 L/min , q = 3.40e − 03 m²/s , W_eff = 5.30e − 02 m , t_acqui = 327.6 s , f_acqui = 25.00Hz , dx = 5.070e − 04 m
2 cameras (PointGreyGrassHooper3), Canal TQH23, 2 obstacles: ACRI10 (0.021 m), ACRI10 (0.005 m), Downstream gate : none
Fr < 1 & U < c_gmin < U_L
Fr < 1 & c_gmin < U < U_L
Fr < 1 & c_gmin < U_L < U
Fr > 1 & U < c_gmin < U_L
Fr > 1 & c_gmin < U < U_L
Fr > 1 & c_gmin < U_L < U



Q = 38.30 L/min , q = 1.20e − 02 m²/s , W_eff = 5.30e − 02 m , t_acqui = 327.6 s , f_acqui = 25.00Hz , dx = 5.070e − 04 m
2 cameras (PointGreyGrassHooper3), Canal TQH23, 2 obstacles: ACRI10 (0.005 m), ACRI10 (0.005 m), Downstream gate : none
Fr < 1 & U < c_gmin < U_L
Fr < 1 & c_gmin < U < U_L
Fr < 1 & c_gmin < U_L < U
Fr > 1 & U < c_gmin < U_L
Fr > 1 & c_gmin < U < U_L
Fr > 1 & c_gmin < U_L < U

In this section, we use the stationary Saint-Venant model to work with analytical expressions of the water depth:

$$\begin{cases} \partial_x (hU) = 0 \\ \partial_x \left(hU^2 + \frac{1}{2} g h^2\right) = -gh\partial_x b \end{cases}$$

This is equivalent to solving the following water level model:

$$\frac{q^2}{2h(x)^2} + g\left(h(x) + b(x)\right) = \frac{U_\infty^2}{2} + gh_\infty$$

We must therefore solve a third-degree polynomial verified by $h(x)$

Boundary conditions:

$$h_\infty = h_{\text{upstream}} = \frac{1}{2}\sqrt[3]{\frac{q^2}{g}} + \frac{b_{\max}}{3} + 2\left(\frac{1}{2}\sqrt[3]{\frac{q^2}{g}} + \frac{b_{\max}}{3}\right)\cos\left(\frac{1}{3}\operatorname{Arccos}\left(1 - \frac{\frac{1}{4}\frac{q^2}{g}}{\left(\frac{1}{2}\sqrt[3]{\frac{q^2}{g}} + \frac{b_{\max}}{3}\right)^3}\right)\right)$$

$$h_{\text{downstream}} = h_{\text{upstream}}\left(\frac{Fr_\infty^2}{4} + \frac{Fr_\infty}{4}\sqrt{Fr_\infty^2 + 8}\right)$$

$$Fr_\infty = Fr_{\text{upstream}} = \frac{U_{\text{upstream}}}{\sqrt{g h_{\text{upstream}}}}$$

$$U_{\text{upstream}} = \frac{Q}{W h_{\text{upstream}}} = \frac{q}{h_{\text{upstream}}}$$

This gives us the following solutions:

- For subcritical regime:

$$h(x) = \frac{h_{\text{sub}}}{3}\left(1 + \frac{Fr_{\text{sub}}^2}{2} - \frac{b(x)}{h_{\text{sub}}}\right) + \frac{2h_{\text{sub}}}{3}\left(1 + \frac{Fr_{\text{sub}}^2}{2} - \frac{b(x)}{h_{\text{sub}}}\right)\cos\left(\frac{1}{3}\text{Arccos}\left(1 - \frac{\frac{27}{4}Fr_{\text{sub}}^2}{\left(1 + \frac{Fr_{\text{sub}}^2}{2} - \frac{b(x)}{h_{\text{sub}}}\right)^3}\right)\right)$$

$$\text{with} \quad h_{\text{sub}} \geqslant h_\infty$$

- For supercritical regime:

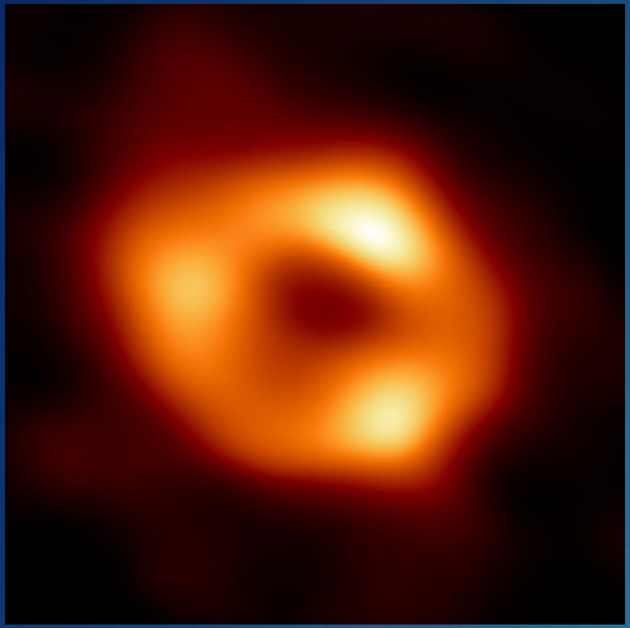
$$h(x) = \frac{\widetilde{h_\infty}}{3}\left(1 + \frac{\widetilde{Fr_\infty}^2}{2} - \frac{b(x)}{\widetilde{h_\infty}}\right) + \frac{2\widetilde{h_\infty}}{3}\left(1 + \frac{\widetilde{Fr_\infty}^2}{2} - \frac{b(x)}{\widetilde{h_\infty}}\right)\cos\left(\frac{1}{3}\text{Arccos}\left(1 - \frac{\frac{27}{4}\widetilde{Fr_\infty}^2}{\left(1 + \frac{\widetilde{Fr_\infty}^2}{2} - \frac{b(x)}{\widetilde{h_\infty}}\right)^3}\right) - \frac{2\pi}{3}\right)$$

$$\text{with} \quad \widetilde{h_\infty} = h_{\text{sup}}\left(\frac{Fr_{\text{sup}}^2}{4} + \frac{Fr_{\text{sup}}}{4}\sqrt{Fr_{\text{sup}}^2 + 8}\right) \quad \text{and} \quad h_{\text{sup}} \leqslant h_\infty\left(\frac{Fr_\infty^2}{4} + \frac{Fr_\infty}{4}\sqrt{Fr_\infty^2 + 8}\right)$$

- For transcritical (accelerating) regime:

$$h(x) = \frac{h_\infty}{3}\left(1 + \frac{Fr_\infty^2}{2} - \frac{b(x)}{h_\infty}\right) + \frac{2h_\infty}{3}\left(1 + \frac{Fr_\infty^2}{2} - \frac{b(x)}{h_\infty}\right)\cos\left(\frac{1}{3}\text{Arccos}\left(1 - \frac{\frac{27}{4}Fr_\infty^2}{\left(1 + \frac{Fr_\infty^2}{2} - \frac{b(x)}{h_\infty}\right)^3}\right) + \frac{2k(x)\pi}{3}\right)$$

$$\text{with} \quad k(x) = \begin{cases} 0 & \text{for } x \in]-\infty; b^{-1}(b_{\max})] \\ -1 & \text{for } x \in [b^{-1}(b_{\max}); +\infty[\end{cases}$$

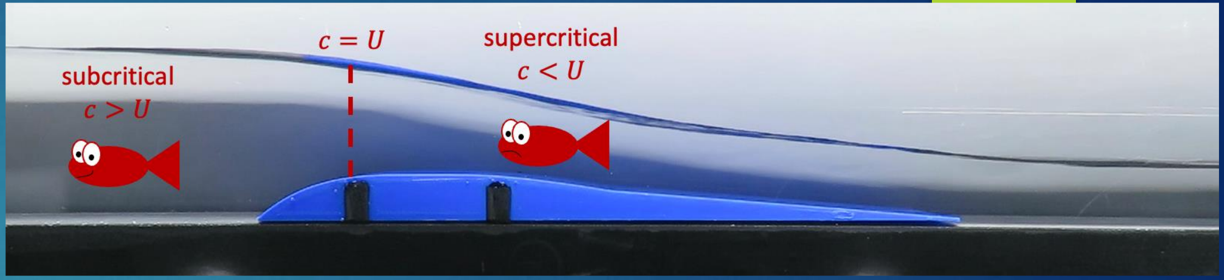


Source: https://Beta.NSF.GOV/EHT

Analogue gravity in interfacial hydrodynamics

William Unruh (1981)[1]

~ analogous to… (kinematically)



$$\mathbf{Fr}_{\mathbf{local}} = \mathbf{1} \Rightarrow \text{Existence of an analogous horizon}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\phi\right) = 0 \text{ with } ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

- $\beta = \frac{v_{\text{escape}}(r)}{c} = \sqrt{\frac{r_s}{r}}$ with $r_s = \frac{2GM}{c^2}$
- $ds^2 = -c^2\left(1 - \frac{v(r)^2}{c^2}\right)dt^2 + 2v(r)dtdr + dr^2 + r^2d\Omega^2$
- $(\omega - v_{\text{escape}}k)^2 = c^2k^2$ with $c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \approx 3\times10^8\ m.s^{-1}$
- $T_{\text{Hawking}} = \frac{\hbar}{4\pi k_B c}\left|\frac{\partial v^2_{\text{escape}}}{\partial r}\right|\Bigg)_{r=\beta^{-1}(1)} = \frac{\hbar c^3}{8\pi k_B GM}$

Stephen Hawking (1974)[2]

$$g^{\mu\nu}k_\mu k_\nu = 0$$

- $Fr = \frac{U(x)}{c} = \frac{U(x)}{\sqrt{gh}}$ with $g = 9.81\ m.s^{-2}$ and h the water depth
- $ds^2 = -c^2\left(1 - \frac{U^2}{c^2}\right)dt^2 + 2Udtdx + dx^2$
- $(\omega - Uk)^2 = c^2k^2$ with no dispersive terms
- $T_{\text{Visser}} = \frac{\hbar}{4\pi k_B}\left|\frac{1}{c}\frac{\partial(c^2 - U^2)}{\partial x}\right|\Bigg)_{x=Fr^{-1}(1)}$

Matt Visser (1998)[3]

White hole: time reversal of the black hole($t \to -t$) ~ White fountain

Euvé et al (2016) [4]

A particular regime : D-E-T_a